AP Calculus AB
Summer Assignments

The attached packet contains mathematical concepts that are necessary to succeed in Calculus.

Material from: Larson, Hostetler, Edwards. *Calculus*, 5th Ed. Heath, 1994.

The following assigned problems will be due the first day of school in September. You must show all your work.

Read each section and then do the problems.

Section 1
#2, 4, 6, 8, 10, 12, 14, 15, 20, 24, 32, 34, 40, 52, 59

Section 2
#3, 8, 16, 22, 27, 28, 29, 30, 34, 40, 44, 54

Section 3
#10, 16, 22, 26, 38, 58, 60

Section 4
#11, 14, 16, 20, 24, 32, 40*, 44, 56
Write equation of indicated line. Ex. 36 is not needed.

Section 5
#2, 9, 18, 20, 34, 42, 44, 54, 56, 67, 68, 69, 70

Section 6
#6, 8, 13, 16, 18, 23, 24, 25, 26, 36, 38, 44, 47, 52, 54, 56, 62, 70, 72

Turn-in completed work to your AP calculus teacher on the <u>FIRST DAY OF SCHOOL</u>.

Remember: ALL WORK MUST BE SHOWN NEATLY ON SEPARATE PIECES OF PAPER – NOT IN THE PACKET.

Real Numbers and the Real Line • Order and Inequalities • Absolute Value and Distance

Real Numbers and the Real Line

Real numbers can be represented by a coordinate system called the **real line** or x-axis (see Figure 1). The real number corresponding to a point on the real line is the **coordinate** of the point. As Figure 1 shows, it is customary to identify those points whose coordinates are integers.

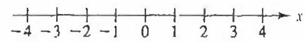


FIGURE 1
The real line

The point on the real line corresponding to zero is the **origin** and is denoted by 0. The **positive direction** (to the right) is denoted by an arrowhead and is the direction of increasing values of x. Numbers to the right of the origin are **positive.** Numbers to the left of the origin are **negative.** The term **nonnegative** describes a number that is positive or zero. The term **nonpositive** describes a number that is negative or zero.

Each point on the real line corresponds to one and only one real number, and each real number corresponds to one and only one point on the real line. This type of relationship is called a **one-to-one correspondence.**

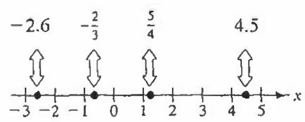


FIGURE 2
Rational numbers

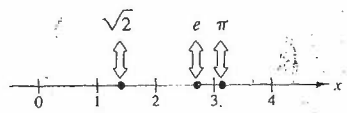


FIGURE 3
Irrational numbers

Each of the four points in Figure 2 corresponds to a **rational number**—one that can be expressed as the ratio of two integers. (Note that $4.5 = \frac{9}{2}$ and $-2.6 = \frac{-13}{5}$.) Rational numbers can be represented either by *terminating decimals* such as $\frac{2}{5} = 0.4$, or by *repeating decimals* such as $\frac{1}{3} = 0.333\ldots = 0.\bar{3}$.

Real numbers that are not rational are **irrational.** Irrational numbers cannot be represented as terminating or repeating decimals. In computations, irrational numbers are represented by decimal approximations. Here are three familiar examples.

$$\sqrt{2} \approx 1.414213562$$
$$\pi \approx 3.141592654$$
$$e \approx 2.718281828$$

(See Figure 3.)

Order and Inequalities

One important property of real numbers is that they can be **ordered.** If a and b are real numbers, then a is **less than** b if $b - a$ is positive. This order is denoted by the **inequality**

$$a < b.$$

The statement b is **greater than** a is equivalent to saying a is less than b. When three real numbers a, b, and c are ordered so that $a < b$ and $b < c$, we say that b is **between** a and c and $a < b < c$.

Geometrically, $a < b$ if and only if a lies to the *left* of b on the real line (see Figure 4). For example, $1 < 2$ because 1 lies to the left of 2 on the real line.

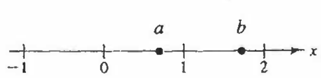


FIGURE 4
$a < b$ if and only if a lies to the left of b.

The following properties are used in working with inequalities. Similar properties are obtained if $<$ is replaced by $\leq$ and $>$ is replaced by $\geq$. (The symbols $\leq$ and $\geq$ mean **less than or equal to** and **greater than or equal to,** respectively.)

Properties of Inequalities

Let a, b, c, d, and k be real numbers.

1. If $a < b$ and $b < c$, then $a < c$. Transitive Property
2. If $a < b$ and $c < d$, then $a + c < b + d$. Add inequalities
3. If $a < b$, then $a + k < b + k$. Add a constant
4. If $a < b$ and $k > 0$, then $ak < bk$. Multiply by a positive constant
5. If $a < b$ and $k < 0$, then $ak > bk$. Multiply by a negative constant

REMARK Note that you *reverse the inequality* when you multiply by a negative number. For example, if $x < 3$, then $-4x > -12$. This also applies to division by a negative number. Thus, if $-2x > 4$, then $x < -2$.

A **set** is a collection of elements. Two common sets are the set of real numbers and the set of points on the real line. Many problems in calculus involve **subsets** of one of these two sets. In such cases it is convenient to use **set notation** of the form $\{x$: condition on $x\}$, which is read as follows.

$$\underbrace{\{}_{\text{The set of all}}\ \underbrace{x}_{x}\ \underbrace{:}_{\text{such that}}\ \underbrace{\text{condition on } x\}}_{\text{a certain condition is true.}}$$

For example, you can describe the set of positive real numbers as

$\{x: 0 < x\}$. Set of positive real numbers

Similarly, you can describe the set of nonnegative real numbers as

$\{x: 0 \leq x\}$. Set of nonnegative real numbers

The **union** of two sets A and B, denoted by $A \cup B$, is the set of elements that are members of *A or B or both.* The **intersection** of two sets A and B, denoted by $A \cap B$, is the set of elements that are members of *A and B.* Two sets are **disjoint** if they have no elements in common.

The most commonly used subsets are **intervals** on the real line. For example, the **open** interval

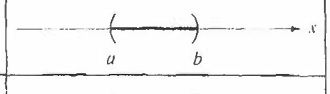
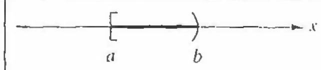
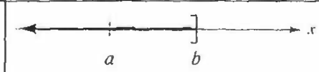
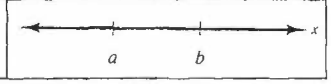
$(a, b) = \{x: a < x < b\}$ Open interval

is the set of all real numbers greater than a and less than b, where a and b are the **endpoints** of the interval. Note that the endpoints are not included in an open interval. Intervals that include their endpoints are **closed** and are denoted by

$[a, b] = \{x: a \leq x \leq b\}$. Closed interval

The nine basic types of intervals on the real line are shown in Table 1. The first four are **bounded intervals** and the remaining five are **unbounded intervals.** Unbounded intervals are also classified as open or closed. The intervals $(-\infty, b)$ and (a, ∞) are open, the intervals $(-\infty, b]$ and $[a, \infty)$ are closed, and the interval $(-\infty, \infty)$ is considered to be both open *and* closed.

TABLE 1
Intervals on the Real Line

	Interval notation	*Set notation*	*Graph*
Bounded open interval	(a, b)	$\{x: a < x < b\}$	
Bounded closed interval	$[a, b]$	$\{x: a \leq x \leq b\}$	
Bounded half-open intervals	$[a, b)$	$\{x: a \leq x < b\}$	
	$(a, b]$	$\{x: a < x \leq b\}$	
Unbounded open intervals	$(-\infty, b)$	$\{x: x < b\}$	
	(a, ∞)	$\{x: a < x\}$	
Unbounded closed intervals	$(-\infty, b]$	$\{x: x \leq b\}$	
	$[a, \infty)$	$\{x: a \leq x\}$	
Entire real line	$(-\infty, \infty)$	$\{x: x \text{ is a real number}\}$	

REMARK The symbols ∞ and $-\infty$ refer to positive and negative infinity. These symbols do not denote real numbers. They simply enable you to describe unbounded conditions more concisely. For instance, the interval $[a, \infty)$ is unbounded to the right because it includes *all* real numbers that are greater than or equal to a.

EXAMPLE 1 Liquid and Gaseous States of Water

Describe the intervals on the real line that correspond to the temperature x (in degrees Celsius) for water in

a. A liquid state **b.** A gaseous state.

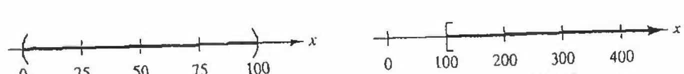
Solution

a. Water is in a liquid state at temperatures greater than 0° and less than 100°, as shown in Figure 5(a).

$$(0, 100) = \{x: 0 < x < 100\}$$

b. Water is in a gaseous state (steam) at temperatures greater than or equal to 100°, as shown in Figure 5(b).

$$[100, \infty) = \{x: 100 \leq x\}$$



(a) Temperature range of water (in degrees Celsius)

(b) Temperature range of steam (in degrees Celsius)

FIGURE 5

A real number a is a **solution** of an inequality if the inequality is **satisfied** (is true) when a is substituted for x. The set of all solutions is the **solution set** of the inequality.

EXAMPLE 2 Solving an Inequality

Solve $2x - 5 < 7$.

Solution

$2x - 5 < 7$	Original inequality
$2x - 5 + 5 < 7 + 5$	Add 5 to both sides
$2x < 12$	Simplify
$\frac{1}{2}(2x) < \frac{1}{2}(12)$	Multiply both sides by $\frac{1}{2}$
$x < 6$	Simplify

The solution set is $(-\infty, 6)$.

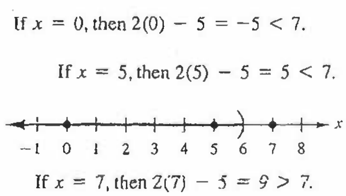


FIGURE 6
Checking solutions of $2x - 5 < 7$.

REMARK In Example 2, all five inequalities listed as steps in the solution are called **equivalent** because they have the same solution set.

Once you have solved an inequality, check some x-values in your solution set to verify that they satisfy the original inequality. You should also check some values outside your solution set to verify that they *do not* satisfy the inequality. For example, Figure 6 shows that when $x = 0$ or $x = 5$ the inequality $2x - 5 < 7$ is satisfied, but when $x = 7$ the inequality $2x - 5 < 7$ is not satisfied.

EXAMPLE 3 Solving a Double Inequality

Solve $-3 \le 2 - 5x \le 12$.

Solution

$-3 \le 2 - 5x \le 12$	Original inequality
$-3 - 2 \le 2 - 5x - 2 \le 12 - 2$	Subtract 2
$-5 \le -5x \le 10$	Simplify
$\frac{-5}{-5} \ge \frac{-5x}{-5} \ge \frac{10}{-5}$	Divide by -5 and reverse both inequalities
$1 \ge x \ge -2$	Simplify

The solution set is $[-2, 1]$, as shown in Figure 7.

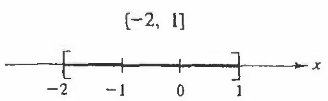


FIGURE 7
Solution set of $-3 \le 2 - 5x \le 12$.

The inequalities in Examples 2 and 3 are **linear inequalities**—that is, they involve first-degree polynomials. To solve inequalities involving polynomials of higher degree, use the fact that a polynomial can change signs *only* at its real **zeros** (the numbers that make the polynomial zero). Between two consecutive real zeros a polynomial must be either entirely positive or entirely negative. This means that when the real zeros of a polynomial are put in order, they divide the real line into **test intervals** in which the polynomial has no sign changes. Thus, if a polynomial has the factored form

$$(x - r_1)(x - r_2) \cdots (x - r_n), \qquad r_1 < r_2 < r_3 < \cdots < r_n$$

then the test intervals are

$$(-\infty, r_1), \quad (r_1, r_2), \ldots, \quad (r_{n-1}, r_n), \quad \text{and} \quad (r_n, \infty).$$

To determine the sign of the polynomial in each test interval, you need to test only *one value* from the interval.

EXAMPLE 4 Solving a Quadratic Inequality

Solve $x^2 < x + 6$.

Solution

$$x^2 < x + 6 \quad \text{Original inequality}$$
$$x^2 - x - 6 < 0 \quad \text{Write in standard form}$$
$$(x - 3)(x + 2) < 0 \quad \text{Factor}$$

The polynomial $x^2 - x - 6$ has $x = -2$ and $x = 3$ as its zeros. Thus, you can solve the inequality by testing the sign of $x^2 - x - 6$ in each of the test intervals $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$. To test an interval, choose any number in the interval and compute the sign of $x^2 - x - 6$. After doing this, you will find that the polynomial is positive for all real numbers in the first and third intervals and negative for all real numbers in the second interval. The solution of the original inequality is therefore $(-2, 3)$, as shown in Figure 8.

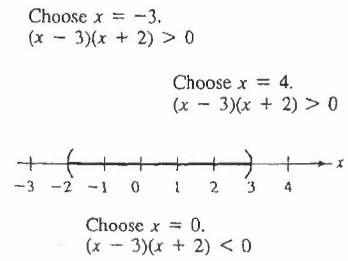


FIGURE 8
Testing an interval.

Absolute Value and Distance

If a is a real number, then the **absolute value** of a is

$$|a| = \begin{cases} a, & \text{if } a \geq 0 \\ -a, & \text{if } a < 0. \end{cases}$$

The absolute value of a number cannot be negative. For example, let $a = -4$. Then, because $-4 < 0$, you have

$$|a| = |-4| = -(-4) = 4.$$

Remember that the symbol $-a$ does not necessarily mean that $-a$ is negative.

Operations with Absolute Value

Let a and b be real numbers and let n be a positive integer.

1. $|ab| = |a|\,|b|$ **2.** $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}, \quad b \neq 0$

3. $|a| = \sqrt{a^2}$ **4.** $|a^n| = |a|^n$

REMARK You are asked to prove these properties in Exercises 65, 67, 68, and 69.

Properties of Inequalities and Absolute Value

Let a and b be real numbers, and let k be a positive real number.

1. $-|a| \leq a \leq |a|$
2. $|a| \leq k$ if and only if $-k \leq a \leq k$.
3. $k \leq |a|$ if and only if $k \leq a$ or $a \leq -k$.
4. *Triangle Inequality:* $|a + b| \leq |a| + |b|$

Properties 2 and 3 are also true if $\leq$ is replaced by $<$.

EXAMPLE 5 Solving an Absolute Value Inequality

Solve $|x - 3| \leq 2$.

Solution Using the second property of inequalities and absolute value, you can rewrite the original inequality as a double inequality.

$$-2 \leq x - 3 \leq 2 \quad \text{Write as double inequality}$$
$$-2 + 3 \leq x - 3 + 3 \leq 2 + 3 \quad \text{Add 3}$$
$$1 \leq x \leq 5 \quad \text{Simplify}$$

The solution set is $[1, 5]$, as shown in Figure 9.

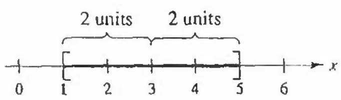


FIGURE 9
Solution set of $|x - 3| \leq 2$.

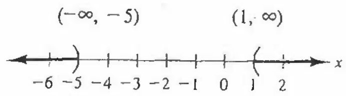


FIGURE 10
Solution set of $3 < |x + 2|$.

EXAMPLE 6 A Two-Interval Solution Set

Solve $3 < |x + 2|$.

Solution Using the third property of inequalities and absolute value, you can rewrite the original inequality as two linear inequalities.

$$3 < x + 2 \quad \text{or} \quad x + 2 < -3$$
$$1 < x \quad \text{or} \quad x < -5$$

The solution set is the union of the disjoint intervals $(-\infty, -5)$ and $(1, \infty)$, as shown in Figure 10.

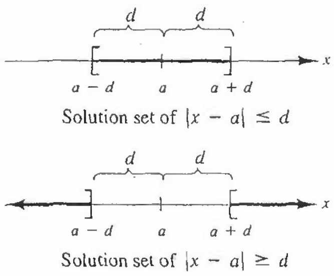


FIGURE 11

Examples 5 and 6 illustrate the general results shown in Figure 11. Note that if $d > 0$, the solution set for the inequality $|x - a| \leq d$ is a *single* interval, whereas the solution set for the inequality $|x - a| \geq d$ is the union of *two* disjoint intervals.

The **distance between two points** a and b on the real line is given by

$$d = |a - b| = |b - a|.$$

The **directed distance from a to b** is $b - a$ and the **directed distance from b to a** is $a - b$, as shown in Figure 12.

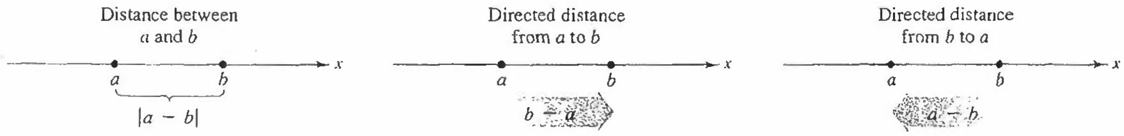


FIGURE 12

EXAMPLE 7 Distance on the Real Line

a. The distance between −3 and 4 is

$$|4-(-3)| = |7| = 7 \quad \text{or} \quad |-3 - 4| = |-7| = 7.$$

(See Figure 13.)

b. The directed distance from −3 to 4 is $4 - (-3) = 7$.

c. The directed distance from 4 to −3 is $-3 - 4 = -7$.

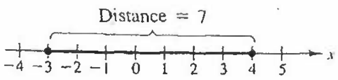


FIGURE 13

The **midpoint** of an interval with endpoints a and b is the average value of a and b. That is,

$$\text{Midpoint of interval } (a, b) = \frac{a + b}{2}.$$

To show that this is the midpoint, you need only show that $(a + b)/2$ is equidistant from a and b.

EXERCISES for Section 1

In Exercises 1–10, determine whether the real number is rational or irrational.

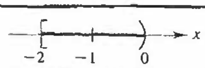
1. 0.7
2. −3678
3. $\frac{3\pi}{2}$
4. $3\sqrt{2} - 1$
5. $4.345\overline{1451}$
6. $\frac{22}{7}$
7. $\sqrt[3]{64}$
8. $0.8177\overline{8177}$
9. $4\frac{5}{8}$
10. $(\sqrt{2})^3$

In Exercises 11–14, express the repeating decimal as a ratio of integers using the following procedure. If $x = 0.6363\ldots$, then $100x = 63.6363\ldots$. Subtracting the first equation from the second produces $99x = 63$ or $x = \frac{63}{99} = \frac{7}{11}$.

11. $0.36\overline{36}$
12. $0.318\overline{18}$
13. $0.297\overline{297}$
14. $0.9900\overline{9900}$

15. Given $a < b$, determine which of the following are true.
 a. $a + 2 < b + 2$
 b. $5b < 5a$
 c. $5 - a > 5 - b$
 d. $\frac{1}{a} < \frac{1}{b}$
 e. $(a - b)(b - a) > 0$
 f. $a^2 < b^2$

16. Complete the table with the appropriate interval notation, set notation, and graph on the real line.

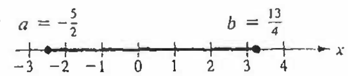
Interval notation	*Set notation*	*Graph*
		
$(-\infty, -4]$		
	$\{x: 3 \le x \le \frac{11}{2}\}$	
$(-1, 7)$		

In Exercises 17–40, solve the inequality and sketch the graph of the solution on the real line.

17. $4x + 1 < 2x$
18. $2x + 7 < 3$
19. $2x - 1 \ge 0$
20. $3x + 1 \ge 2x + 2$
21. $-4 < 2x - 3 < 4$
22. $0 \le x + 3 < 5$
23. $\frac{3}{4}x > x + 1$
24. $-1 < -\frac{x}{3} < 1$
25. $\frac{x}{2} + \frac{x}{3} > 5$
26. $x > \frac{1}{x}$
27. $|x| < 1$
28. $\frac{x}{2} - \frac{x}{3} > 5$
29. $\left|\frac{x - 3}{2}\right| \ge 5$
30. $\left|\frac{x}{2}\right| > 3$
31. $|x - a| < b,\ b > 0$
32. $|x + 2| < 5$
33. $|2x + 1| < 5$
34. $|3x + 1| \ge 4$
35. $|1 - \frac{2}{3}x| < 1$
36. $|9 - 2x| < 1$
37. $x^2 \le 3 - 2x$
38. $x^4 - x \le 0$
39. $x^2 + x - 1 \le 5$
40. $2x^2 + 1 < 9x - 3$

In Exercises 41–44, find the directed distance from a to b, the directed distance from b to a, and the distance between a and b.

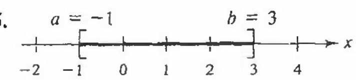
41. 

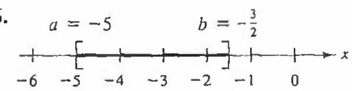
42. 

43. a. $a = 126,\ b = 75$
 b. $a = -126,\ b = -75$

44. a. $a = 9.34,\ b = -5.65$
 b. $a = \frac{16}{5},\ b = \frac{112}{75}$

In Exercises 45–48, find the midpoint of the interval.

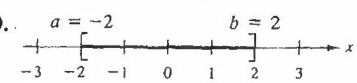
45. 

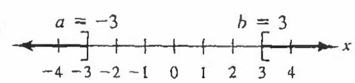
46. 

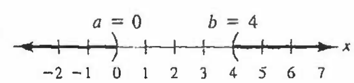
47. a. $[7, 21]$
 b. $[8.6, 11.4]$

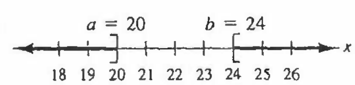
48. a. $[-6.85, 9.35]$
 b. $[-4.6, -1.3]$

In Exercises 49–54, use absolute values to define the interval (or pair of intervals) on the real line.

49. 

50. 

51. 

52. 

59. *Fair Coin* To determine whether a coin is fair (has an equal probability of landing tails up or heads up), an experimenter tosses it 100 times and records the number of heads x. The coin is declared unfair if

$$\left|\frac{x - 50}{5}\right| \ge 1.645.$$

For what values of x will the coin be declared unfair?

Suppose you want to determine the distance d between the two points (x_1, y_1) and (x_2, y_2) in the plane. If the points lie on a horizontal line, then $y_1 = y_2$ and the distance between the points is $|x_2 - x_1|$. If the points lie on a vertical line, then $x_1 = x_2$ and the distance between the points is $|y_2 - y_1|$. If the two points do not lie on a horizontal or vertical line, then they can be used to form a right triangle, as shown in Figure 17. The length of the vertical side of the triangle is $|y_2 - y_1|$, and the length of the horizontal side is $|x_2 - x_1|$. By the Pythagorean Theorem, it follows that

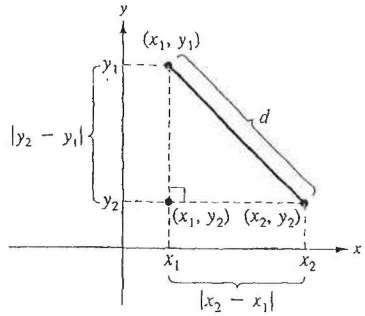


FIGURE 17
The distance between two points.

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$
$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}.$$

Replacing $|x_2 - x_1|^2$ and $|y_2 - y_1|^2$ by the equivalent expressions $(x_2 - x_1)^2$ and $(y_2 - y_1)^2$ produces the following result.

Distance Formula

The distance d between the points (x_1, y_1) and (x_2, y_2) in the plane is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

EXAMPLE 1 Finding the Distance Between Two Points

Find the distance between the points $(-2, 1)$ and $(3, 4)$.

Solution

$$\begin{aligned} d &= \sqrt{[3 - (-2)]^2 + (4 - 1)^2} && \text{Distance Formula} \\ &= \sqrt{(5)^2 + (3)^2} \\ &= \sqrt{25 + 9} \\ &= \sqrt{34} \\ &\approx 5.83 \end{aligned}$$

EXAMPLE 2 Verifying a Right Triangle

Verify that the points $(2, 1)$, $(4, 0)$, and $(5, 7)$ form the vertices of a right triangle

Solution Figure 18 shows the triangle formed by the three points. The lengths of the three sides are as follows.

$$d_1 = \sqrt{(5 - 2)^2 + (7 - 1)^2} = \sqrt{9 + 36} = \sqrt{45}$$
$$d_2 = \sqrt{(4 - 2)^2 + (0 - 1)^2} = \sqrt{4 + 1} = \sqrt{5}$$
$$d_3 = \sqrt{(5 - 4)^2 + (7 - 0)^2} = \sqrt{1 + 49} = \sqrt{50}$$

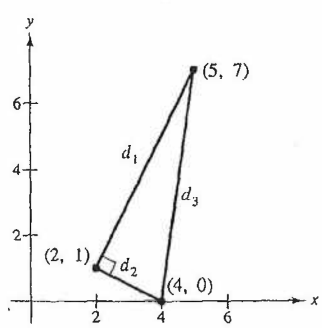


FIGURE 18
Verifying a right triangle.

Because

$$d_1^2 + d_2^2 = 45 + 5 = 50 \qquad \text{Sum of squares of sides}$$

and

$$d_3^2 = 50 \qquad \text{Square of hypotenuse}$$

you can apply the Pythagorean Theorem to conclude that the triangle is a right triangle.

SECTION The Cartesian Plane

2 The Cartesian Plane • The Distance and Midpoint Formulas • Equations of Circles

The Cartesian Plane

An **ordered pair** (x, y) of real numbers has x as its *first* member and y as its *second* member. The model for representing ordered pairs is called the **rectangular coordinate system,** or the **Cartesian plane,** after the French mathematician René Descartes (1596–1650). It is developed by considering two real lines intersecting at right angles (see Figure 14).

The horizontal real line is usually called the ***x*-axis,** and the vertical real line is usually called the ***y*-axis.** Their point of intersection is the **origin.** The two axes divide the plane into four **quadrants.**

RENÉ DESCARTES

Descartes made many contributions to philosophy, science, and mathematics. The idea of representing points in the plane by pairs of real numbers and representing curves in the plane by equations was described by Descartes in his book titled *La Géométrie*, published in 1637.

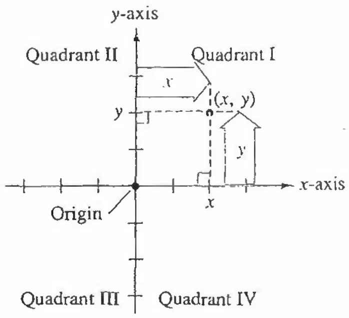


FIGURE 14
The Cartesian plane

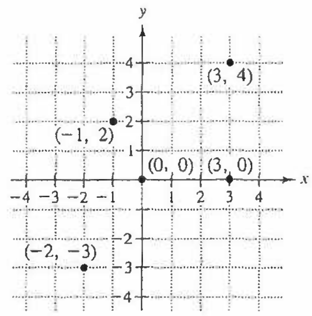


FIGURE 15
Points represented by ordered pairs.

Each point in the plane is identified by an ordered pair (x, y) of real numbers x and y, called **coordinates** of the point. The number x represents the directed distance from the y-axis to the point, and the number y represents the directed distance from the x-axis to the point (see Figure 14). For the point (x, y), the first coordinate is the x-coordinate or **abscissa,** and the second coordinate is the y-coordinate or **ordinate.** For example, Figure 15 shows the locations of the points $(-1, 2)$, $(3, 4)$, $(0, 0)$, $(3, 0)$, and $(-2, -3)$ in the Cartesian plane.

REMARK The signs of the coordinates of a point determine the quadrant in which the point lies. For instance, if $x > 0$ and $y < 0$, then (x, y) lies in Quadrant IV.

Note that an ordered pair (a, b) is used to denote either a point in the plane *or* an open interval on the real line. This, however, should not be confusing—the nature of the problem should clarify whether a point in the plane or an open interval is being discussed.

TECHNOLOGY Note that Figure 15 shows only a small portion of the Cartesian plane—the portion for which $-4 \le x \le 4$ and $-4 \le y \le 4$. Such a portion is called a **viewing rectangle.** If you are using a graphing utility in this course, you need to become familiar with the steps required to change the viewing rectangle on your graphing utility.

The Distance and Midpoint Formulas

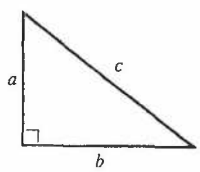


FIGURE 16
The Pythagorean Theorem: $a^2 + b^2 = c^2$.

Recall from the Pythagorean Theorem that, in a right triangle, the hypotenuse c and sides a and b are related by $a^2 + b^2 = c^2$. Conversely, if $a^2 + b^2 = c^2$, then the triangle is a right triangle (see Figure 16).

EXAMPLE 3 Using the Distance Formula

Find x so that the distance between $(x, 3)$ and $(2, -1)$ is 5.

Solution Using the Distance Formula, you can write the following.

$$5 = \sqrt{(x-2)^2 + [3-(-1)]^2} \quad \text{Distance Formula}$$
$$25 = (x^2 - 4x + 4) + 16 \quad \text{Square both sides}$$
$$0 = x^2 - 4x - 5 \quad \text{Write in standard form}$$
$$0 = (x-5)(x+1) \quad \text{Factor}$$

Therefore, $x = 5$ or $x = -1$, and you can conclude that there are two solutions. That is, each of the points $(5, 3)$ and $(-1, 3)$ lies 5 units from the point $(2, -1)$, as shown in Figure 19.

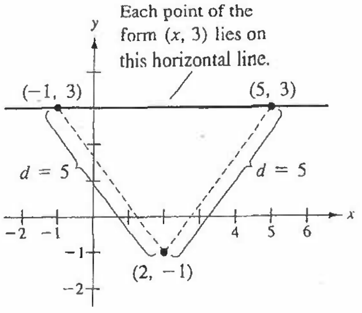


FIGURE 19
Given a distance, find a point.

The coordinates of the **midpoint** of the line segment joining two points can be found by "averaging" the x-coordinates of the two points and "averaging" the y-coordinates of the two points. That is, the midpoint of the line segment joining the points (x_1, y_1) and (x_2, y_2) in the plane is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right). \quad \text{Midpoint Formula}$$

For instance, the midpoint of the line segment joining the points $(-5, -3)$ and $(9, 3)$ is

$$\left(\frac{-5+9}{2}, \frac{-3+3}{2}\right) = (2, 0)$$

as shown in Figure 20.

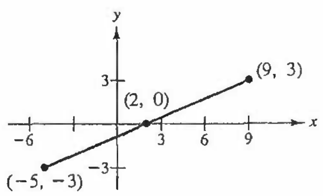


FIGURE 20
Midpoint of a line segment.

Equations of Circles

A **circle** can be defined as the set of all points in a plane that are equidistant from a fixed point. The fixed point is the **center** of the circle, and the distance between the center and a point on the circle is the **radius** (see Figure 21).

You can use the Distance Formula to write an equation for the circle with center (h, k) and radius r. Let (x, y) be any point on the circle. Then the distance between (x, y) and the center (h, k) is given by

$$\sqrt{(x-h)^2 + (y-k)^2} = r.$$

By squaring both sides of this equation, you obtain the **standard form of the equation of a circle.**

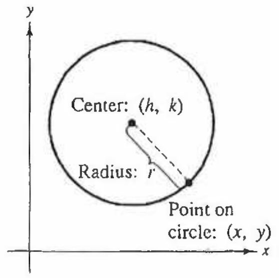


FIGURE 21
Definition of a circle

FOR FURTHER INFORMATION Can you recognize the graph of the equation

$$x^4 + 2x^2y^2 + y^4 - 5x^2 - 5y^2 = -4?$$

For the solution, see the article "Single Equations Can Draw Pictures" by Keith M. Kendig in the March, 1991 issue of *The College Mathematics Journal.*

Standard Form of the Equation of a Circle

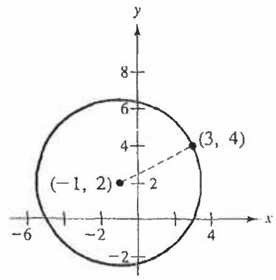
The point (x, y) lies on the circle of radius r and center (h, k) if and only if

$$(x-h)^2 + (y-k)^2 = r^2.$$

The standard form of the equation of a circle with center at the origin, $(h, k) = (0, 0)$, is

$$x^2 + y^2 = r^2.$$

If $r = 1$, then the circle is called the **unit circle.**



$(x + 1)^2 + (y - 2)^2 = 20$

FIGURE 22
Standard form of the equation of a circle.

EXAMPLE 4 Finding the Equation of a Circle

The point (3, 4) lies on a circle whose center is at (−1, 2), as shown in Figure 22. Find an equation for the circle.

Solution The radius of the circle is the distance between (−1, 2) and (3, 4).

$$r = \sqrt{[3 - (-1)]^2 + (4 - 2)^2} = \sqrt{16 + 4} = \sqrt{20}$$

You can write the standard form of the equation of this circle as

$$[x - (-1)]^2 + (y - 2)^2 = (\sqrt{20})^2$$

$$(x + 1)^2 + (y - 2)^2 = 20. \qquad \text{Standard form}$$

By squaring and simplifying, the equation $(x - h)^2 + (y - k)^2 = r^2$ can be written in the following **general form of the equation of a circle.**

$$Ax^2 + Ay^2 + Cx + Dy + F = 0, \qquad A \neq 0$$

To convert such an equation to the standard form

$$(x - h)^2 + (y - k)^2 = p$$

you can use a process called **completing the square.** If $p > 0$, then the graph of the equation is a circle. If $p = 0$, then the graph is the single point (h, k). If $p < 0$, then the equation has no graph.

EXAMPLE 5 Completing the Square

Sketch the graph of the circle whose general equation is

$$4x^2 + 4y^2 + 20x - 16y + 37 = 0.$$

Solution To complete the square, first divide by 4 so that the coefficients of x^2 and y^2 are both 1.

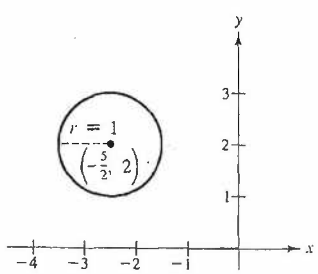
$$4x^2 + 4y^2 + 20x - 16y + 37 = 0 \qquad \text{General form}$$

$$x^2 + y^2 + 5x - 4y + \frac{37}{4} = 0 \qquad \text{Divide by 4}$$

$$(x^2 + 5x + \quad) + (y^2 - 4y + \quad) = -\frac{37}{4} \qquad \text{Group terms}$$

$$\left(x^2 + 5x + \underbrace{\frac{25}{4}}_{(\text{half})^2}\right) + (y^2 - 4y + \underbrace{4}_{(\text{half})^2}) = -\frac{37}{4} + \frac{25}{4} + 4 \qquad \text{Complete the square by adding } \frac{25}{4} \text{ and 4 to both sides}$$

$$\left(x + \frac{5}{2}\right)^2 + (y - 2)^2 = 1 \qquad \text{Standard form}$$



$\left(x + \frac{5}{2}\right)^2 + (y - 2)^2 = 1$

FIGURE 23
A circle with a radius of 1 and center at $(-\frac{5}{2}, 2)$.

Note that you complete the square by adding the square of half the coefficient of x *and* the square of half the coefficient of y to both sides of the equation. The circle is centered at $(-\frac{5}{2}, 2)$ and its radius is 1, as shown in Figure 23.

EXERCISES for Section 2

In Exercises 1–6, (a) plot the points, (b) find the distance between the points, and (c) find the midpoint of the line segment joining the points.

1. $(2, 1), (4, 5)$
2. $(-3, 2), (3, -2)$
3. $(\frac{1}{2}, 1), (-\frac{3}{2}, -5)$
4. $(\frac{2}{3}, -\frac{1}{3}), (\frac{5}{6}, 1)$
5. $(1, \sqrt{3}), (-1, 1)$
6. $(-2, 0), (0, \sqrt{2})$

In Exercises 7–10, show that the points are the vertices of the polygon. (A rhombus is a quadrilateral whose sides are all of the same length.)

	Vertices	*Figure*
7.	$(4, 0), (2, 1), (-1, -5)$	Right triangle
8.	$(1, -3), (3, 2), (-2, 4)$	Isosceles triangle
9.	$(0, 0), (1, 2), (2, 1), (3, 3)$	Rhombus
10.	$(0, 1), (3, 7), (4, 4), (1, -2)$	Parallelogram

11. *Federal Debt* The table lists the per capita federal debt for the United States from 1950 to 1990. (*Source:* U. S. Treasury Department)

Year	1950	1960	1970
Per capita debt	\$1688	\$1572	\$1807

Year	1980	1985	1990
Per capita debt	\$3981	\$7614	\$12,848

Select reasonable scales on the coordinate axes and plot the points (x, y) where y is the per capita debt and x is the time in years, with $x = 0$ corresponding to 1950.

12. *Life Expectancy* The table gives the life expectancy of a child (at birth) from 1920 to 1990. (*Source:* Department of Health and Human Services)

Year	1920	1930	1940	1950
Life expectancy	54.1	59.7	62.9	68.2

Year	1960	1970	1980	1990
Life expectancy	69.7	70.8	73.7	75.2

Select reasonable scales on the coordinate axes and plot the points (x, y) where y is the life expectancy and x is the time in years, with $x = 0$ corresponding to 1950.

In Exercises 13–16, use the Distance Formula to determine whether the points lie on the same line.

13. $(0, -4), (2, 0), (3, 2)$
14. $(0, 4), (7, -6), (-5, 11)$
15. $(-2, 1), (-1, 0), (2, -2)$
16. $(-1, 1), (3, 3), (5, 5)$

In Exercises 17 and 18, find x so that the distance between the points is 5.

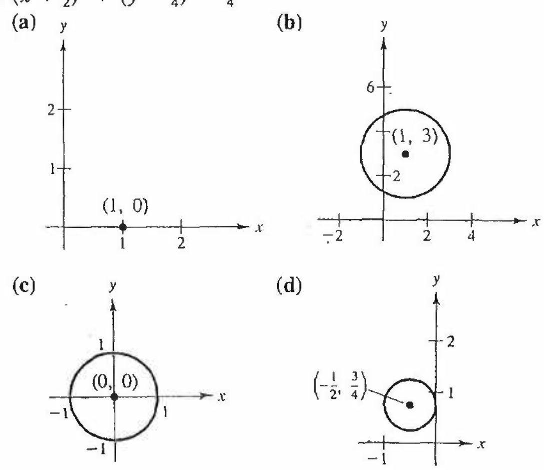
17. $(0, 0), (x, -4)$
18. $(2, -1), (x, 2)$

In Exercises 21 and 22, find the relationship between x and y so that (x, y) is equidistant from the two given points.

21. $(4, -1), (-2, 3)$
22. $(3, \frac{5}{2}), (-7, -1)$

In Exercises 27–30, match the equation with its graph. [Graphs are labeled (a), (b), (c), and (d).]

27. $x^2 + y^2 = 1$
28. $(x - 1)^2 + (y - 3)^2 = 4$
29. $(x - 1)^2 + y^2 = 0$
30. $(x + \frac{1}{2})^2 + (y - \frac{3}{4})^2 = \frac{1}{4}$



In Exercises 31–40, write the equation of the circle in general form.

31. Center: $(0, 0)$
Radius: 3
32. Center: $(0, 0)$
Radius: 5
33. Center: $(2, -1)$
Radius: 4
34. Center: $(-4, 3)$
Radius: $\frac{5}{8}$
35. Center: $(-1, 2)$
Point on circle: $(0, 0)$
36. Center: $(3, -2)$
Point on circle: $(-1, 1)$
37. Endpoints of diameter: $(2, 5), (4, -1)$
38. Endpoints of diameter: $(1, 1), (-1, -1)$
39. Points on circle: $(0, 0), (0, 8), (6, 0)$
40. Points on circle: $(1, -1), (2, -2), (0, -2)$

In Exercises 43–50, write the equation of the circle in standard form and sketch its graph.

43. $x^2 + y^2 - 2x + 6y + 6 = 0$
44. $x^2 + y^2 - 2x + 6y - 15 = 0$
45. $x^2 + y^2 - 2x + 6y + 10 = 0$
46. $3x^2 + 3y^2 - 6y - 1 = 0$
47. $2x^2 + 2y^2 - 2x - 2y - 3 = 0$
48. $4x^2 + 4y^2 - 4x + 2y - 1 = 0$
49. $16x^2 + 16y^2 + 16x + 40y - 7 = 0$
50. $x^2 + y^2 - 4x + 2y + 3 = 0$

In Exercises 53 and 54, sketch the set of all points satisfying the inequality.

53. $x^2 + y^2 - 4x + 2y + 1 \le 0$
54. $(x - 1)^2 + (y - \frac{1}{2})^2 > 1$

SECTION 3

Graphs of Equations

The Graph of an Equation • Intercepts of a Graph • Symmetry of a Graph • Points of Intersection • Mathematical Models

The Graph of an Equation

Consider the equation $3x + y = 7$. The point (2, 1) is a **solution point** of the equation because the equation is satisfied (is true) when 2 is substituted for x and 1 is substituted for y. This equation has many other solutions, such as (1, 4) and (0, 7). To systematically find other solutions, solve the original equation for y.

$$y = 7 - 3x \qquad \text{Solve original equation for } y$$

Then construct a **table of values** by substituting several values of x, as shown in Table 2. From the table, you can see that (0, 7), (1, 4), (2, 1), (3, −2), and (4, −5) are solutions of the original equation

$$3x + y = 7. \qquad \text{Original equation}$$

TABLE 2

x	0	1	2	3	4
y	7	4	1	−2	−5

Like many equations, this equation has an infinite number of solutions. The set of all solution points is the **graph** of the equation, as shown in Figure 24.

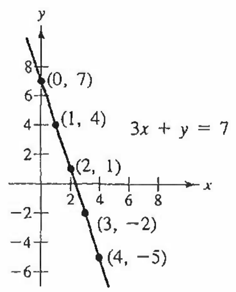


FIGURE 24
Solution points of $3x + y = 7$.

REMARK Even though we refer to the sketch shown in Figure 24 as the graph of $3x + y = 7$, it really represents only a *portion* of the graph. The entire graph would extend beyond the page.

In this course, you will study many sketching aids. The simplest is point plotting—that is, plotting enough points so that the basic shape of the graph becomes apparent.

EXAMPLE 1 Sketching a Graph by Point Plotting

Sketch the graph of $y = x^2 - 2$.

Solution First, construct a table of values, as shown in Table 3.

TABLE 3

x	−2	−1	0	1	2	3
y	2	−1	−2	−1	2	7

Next, plot the points given by the table. Finally, connect the points by a *smooth curve,* as shown in Figure 25. This particular graph is a **parabola.** It is one of the conic sections you will study in Chapter 9.

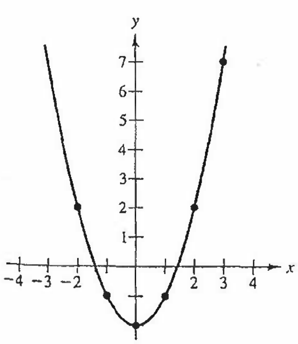


FIGURE 25
The parabola: $y = x^2 - 2$

One disadvantage of point plotting is that to get a good idea about the shape of a graph, you may need to plot many points. With only a few points, you could badly misrepresent the graph. For instance, suppose that to sketch the graph of

$$y = \frac{1}{30}x(39 - 10x^2 + x^4)$$

you plotted only five points: (−3, −3), (−1, −1), (0, 0), (1, 1), and (3, 3) as shown in Figure 26(a). From these five points, you might conclude that the graph is a line. This, however, is not correct. By plotting several more points, you can see that the graph is more complicated, as shown in Figure 26(b).

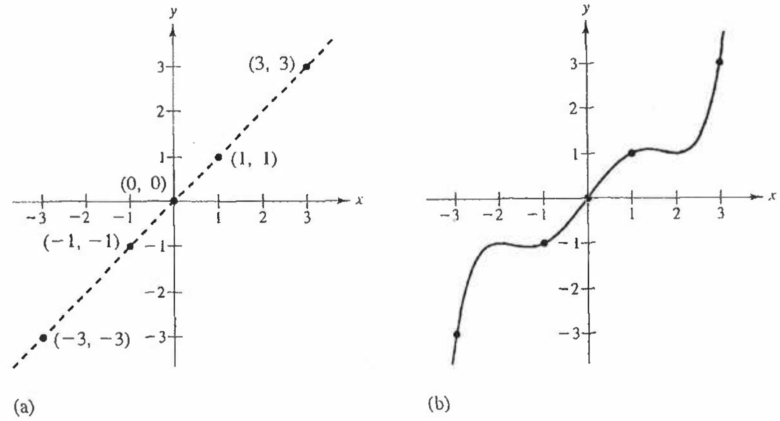


FIGURE 26
Plotting only a few points can misrepresent a graph.

TECHNOLOGY Modern technology has made sketching graphs easier. Even with technology, however, it is possible to badly misrepresent a graph. For instance, each of the graphing utility screens shown in Figure 27 shows a portion of the graph of

$$y = x^3 - x^2 - 25.$$

From the screen on the left, you might assume that the graph is a line. From the screen on the right, however, you can see that the graph is not a line. Thus, whether you are sketching a graph by hand or using a graphing utility, you must realize that different "viewing rectangles" can produce very different views of a graph. In choosing a viewing rectangle, your goal is to show a view of the graph that fits well in the context of the problem.

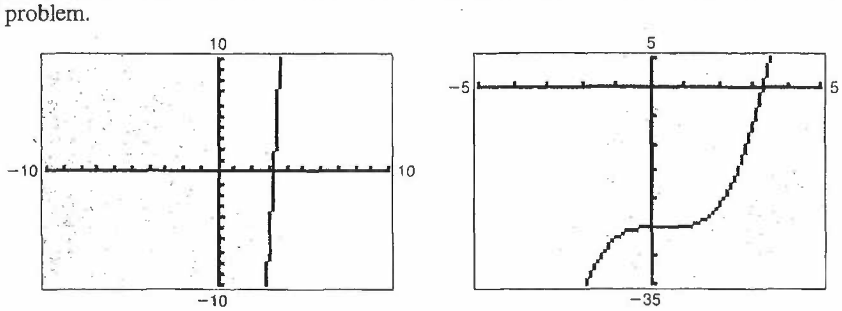


FIGURE 27
Graphing utility screens of $y = x^3 - x^2 - 25$.

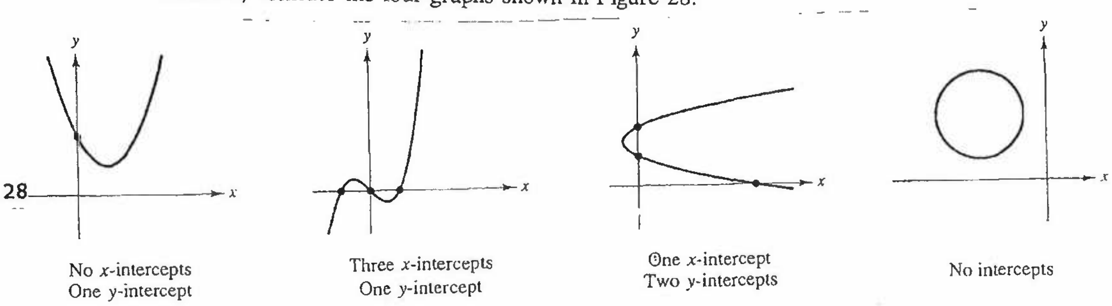
Intercepts of a Graph

Two types of solution points that are especially useful are those having zero as their x-coordinate or y-coordinate. Such points are called **intercepts** because they are the points at which the graph intersects the x-axis or y-axis. The point $(a, 0)$ is an ***x*-intercept** of the graph of an equation if it is a solution point of the equation. To find the x-intercepts of a graph, let y be zero and solve the equation for x. The point $(0, b)$ is a ***y*-intercept** of the graph of an equation if it is a solution point of the equation. To find the y-intercepts of a graph, let x be zero and solve the equation for y.

REMARK Some texts denote the x-intercept as the x-coordinate of the point $(a, 0)$ rather than the point itself. Unless it is necessary to make a distinction, we will use the term *intercept* to mean either the point or the coordinate.

It is possible for a graph to have no intercepts, or it might have several. For instance, consider the four graphs shown in Figure 28.

FIGURE 28



EXAMPLE 2 Finding x- and y-Intercepts

Find the x- and y-intercepts of the graph of $y = x^3 - 4x$.

Solution To find the x-intercepts, let y be zero and solve for x.

$$x^3 - 4x = 0 \qquad \text{Let } y \text{ be zero}$$

$$x(x - 2)(x + 2) = 0 \qquad \text{Factor}$$

$$x = 0, 2, \text{ or } -2 \qquad \text{Solve for } x$$

Because this equation has three solutions, you can conclude that the graph has three x-intercepts:

$$(0, 0), (2, 0), \text{ and } (-2, 0). \qquad x\text{-intercepts}$$

To find the y-intercepts, let x be zero. Doing this produces $y = 0$. Thus, the y-intercept is

$$(0, 0). \qquad y\text{-intercept}$$

(See Figure 29.)

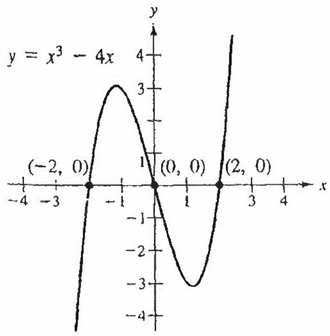


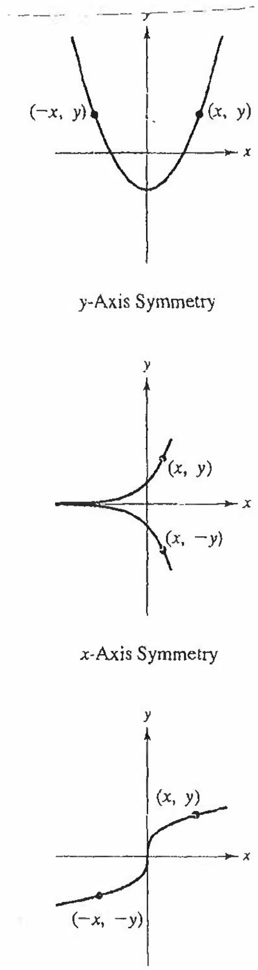
FIGURE 29
Intercepts

Symmetry of a Graph

The following three types of symmetry can be used to help sketch the graph of an equation (see Figure 30).

1. A graph is **symmetric with respect to the y-axis** if, whenever (x, y) is a point on the graph, $(-x, y)$ is also a point on the graph. This means that the portion of the graph to the left of the y-axis is a mirror image of the portion to the right of the y-axis.
2. A graph is **symmetric with respect to the x-axis** if, whenever (x, y) is a point on the graph, $(x, -y)$ is also a point on the graph. This means that the portion of the graph above the x-axis is a mirror image of the portion below the x-axis.
3. A graph is **symmetric with respect to the origin** if, whenever (x, y) is a point on the graph, $(-x, -y)$ is also a point on the graph. This means that the graph is unchanged by a rotation of 180° about the origin.

Knowing that a graph has symmetry *before* attempting to sketch it is helpful because it reduces the number of solution points you need to find.



Origin Symmetry
FIGURE 30

Tests for Symmetry

1. The graph of an equation in x and y is symmetric with respect to the y-axis if replacing x by $-x$ yields an equivalent equation.
2. The graph of an equation in x and y is symmetric with respect to the x-axis if replacing y by $-y$ yields an equivalent equation.
3. The graph of an equation in x and y is symmetric with respect to the origin if replacing x by $-x$ and y by $-y$ yields an equivalent equation.

EXAMPLE 3 Testing for Origin Symmetry

Show that the graph of

$$y = 2x^3 - x$$

is symmetric with respect to the origin.

Solution

$$y = 2x^3 - x \qquad \text{Original equation}$$

$$-y = 2(-x)^3 - (-x) \qquad \text{Replace } x \text{ by } -x \text{ and } y \text{ by } -y$$

$$-y = -2x^3 + x \qquad \text{Simplify}$$

$$y = 2x^3 - x \qquad \text{Equivalent equation}$$

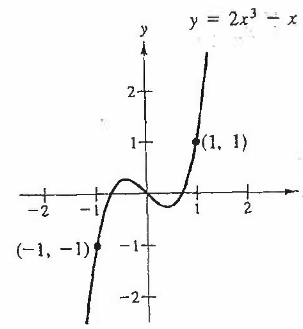


FIGURE 31
Origin symmetry

Because the replacement produces an equivalent equation, you can conclude that the graph of $y = 2x^3 - x$ is symmetric with respect to the origin, as shown in Figure 31.

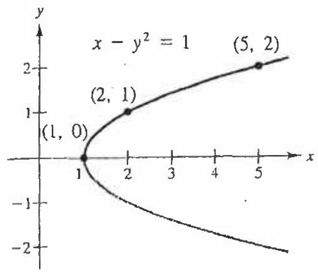


FIGURE 32
First, plot the points above the x-axis, then use symmetry to complete the graph.

EXAMPLE 4 Using Symmetry to Sketch a Graph

Sketch the graph of $x - y^2 = 1$.

Solution The graph is symmetric with respect to the x-axis because replacing y by $-y$ yields an equivalent equation.

$$x - y^2 = 1 \quad \text{Original equation}$$
$$x - (-y)^2 = 1 \quad \text{Replace } y \text{ by } -y$$
$$x - y^2 = 1 \quad \text{Equivalent equation}$$

This means that the portion of the graph below the x-axis is a mirror image of the portion above the x-axis. To sketch the graph, first sketch the portion above the x-axis. Then reflect in the x-axis to obtain the entire graph, as shown in Figure 32.

Points of Intersection

A **point of intersection** of the graphs of two equations is a point that satisfies both equations. You can find the points of intersection of two graphs by solving their equations simultaneously.

EXAMPLE 5 Finding Points of Intersection

Find all points of intersection of the graphs of

$$x^2 - y = 3 \quad \text{and} \quad x - y = 1.$$

Solution It is helpful to begin by sketching the graphs of both equations on the *same* coordinate plane, as shown in Figure 33. When this is done, it appears that the two graphs have two points of intersection. To find these two points, you can use the following steps.

$$y = x^2 - 3 \quad \text{Solve first equation for } y$$
$$y = x - 1 \quad \text{Solve second equation for } y$$
$$x^2 - 3 = x - 1 \quad \text{Equate } y\text{-values}$$
$$x^2 - x - 2 = 0 \quad \text{Write in standard form}$$
$$(x - 2)(x + 1) = 0 \quad \text{Factor}$$
$$x = 2 \text{ or } -1 \quad \text{Solve for } x$$

The corresponding values of y are obtained by substituting $x = 2$ and $x = -1$ into either of the original equations. Doing this produces two points of intersection:

$$(2, 1) \quad \text{and} \quad (-1, -2). \quad \text{Points of intersection}$$

You can check these points by substituting into *both* of the original equations.

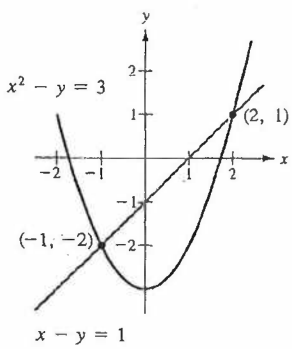


FIGURE 33
Two points of intersection.

Mathematical Models

Real-life applications of mathematics often use equations as **mathematical models.** In developing a mathematical model to represent actual data, you should strive for two (often conflicting) goals: accuracy and simplicity. That is, you want the model to be simple enough to be workable, yet accurate enough to produce meaningful results.

The increasing concentration of carbon dioxide in the earth's atmosphere has been measured at the Mauna Loa Observatory in Hawaii.

EXAMPLE 6 The Rise in Atmospheric Carbon Dioxide

Between 1960 and 1990, the Mauna Loa Observatory in Hawaii recorded the carbon dioxide concentration y (in parts per million) in the earth's atmosphere. The January readings for each year are shown in Figures 34 and 35. In Figure 34, a linear model,

$$y = 313.6 + 1.24t, \qquad \text{Linear model}$$

where $t = 0$ represents 1960, has been fit to the data. In Figure 35, a quadratic model,

$$y = 316.2 + 0.70t + 0.018t^2, \qquad \text{Quadratic model}$$

has been fit to the data. Which model better represents the data? In the July, 1990 issue of *Scientific American,* these data were used to predict the carbon dioxide level in the earth's atmosphere in the year 2035. The prediction was 470 parts per million. Which, if either, of the two models could have been used to make this prediction?

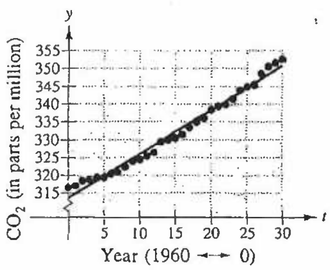


FIGURE 34

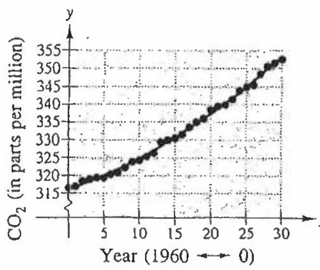


FIGURE 35

Solution To answer the first question, you need to define what "better" means. When a statistical definition involving the squares of the differences between the actual y-values and the y-values given by the model is used, the quadratic model is better. This conclusion is shared by the author of the *Scientific American* article. If the linear model with $t = 75$ (for 2035) were used, the prediction would be

$$y = 313.6 + 1.24(75) = 406.6. \qquad \text{Linear model}$$

If the quadratic model were used, the prediction for 2035 would be

$$y = 316.2 + 0.70(75) + 0.018(75)^2 = 469.95. \qquad \text{Quadratic model}$$

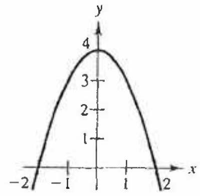
REMARK These models were developed using a procedure called least squares regression (see Section 13.9). The linear model has a correlation given by $r^2 = 0.984$. The quadratic model has a correlation given by $r^2 = 0.997$.

EXERCISES for Section 3

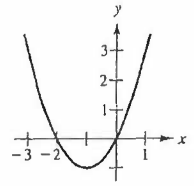
In Exercises 1–6, match the equation with its graph. [Graphs are labeled (a), (b), (c), (d), (e), and (f).]

1. $y = x - 2$
2. $y = -\frac{1}{2}x + 2$
3. $y = x^2 + 2x$
4. $y = \sqrt{9 - x^2}$
5. $y = 4 - x^2$
6. $y = x^3 - x$

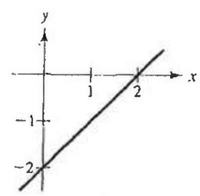
(a)



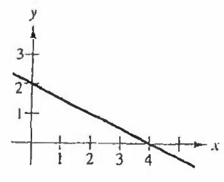
(b)



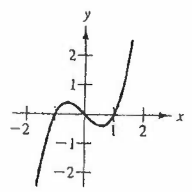
(c)



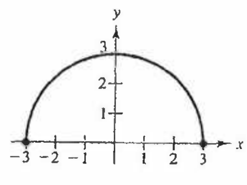
(d)



(e)



(f)



In Exercises 7–16, find the intercepts.

7. $y = 2x - 3$
8. $y = (x - 1)(x - 3)$
9. $y = x^2 + x - 2$
10. $y^2 = x^3 - 4x$
11. $y = x^2\sqrt{9 - x^2}$
12. $xy = 4$
13. $y = \dfrac{x - 1}{x - 2}$
14. $y = \dfrac{x^2 + 3x}{(3x + 1)^2}$
15. $x^2y - x^2 + 4y = 0$
16. $y = 2x - \sqrt{x^2 + 1}$

In Exercises 17–26, check for symmetry with respect to both axes and to the origin.

17. $y = x^2 - 2$
18. $y = x^4 - x^2 + 3$
19. $x^2y - x^2 + 4y = 0$
20. $xy - \sqrt{4 - x^2} = 0$
21. $y^2 = x^3 - 4x$
22. $xy^2 = -10$
23. $y = x^3 + x$
24. $xy = 1$
25. $y = \dfrac{x}{x^2 + 1}$
26. $y = x^3 + x - 3$

In Exercises 27–30, determine whether the points lie on the graph of the equation.

27. Equation: $2x - y - 3 = 0$
Points: $(1, 2)$, $(1, -1)$, $(4, 5)$

28. Equation: $x^2 + y^2 = 4$
Points: $(1, -\sqrt{3})$, $(\frac{1}{2}, -1)$, $(\frac{3}{2}, \frac{7}{2})$

29. Equation: $x^2y - x^2 + 4y = 0$
Points: $(1, \frac{1}{5})$, $(2, \frac{1}{2})$, $(-1, -2)$

30. Equation: $x^2 - xy + 4y = 3$
Points: $(0, 2)$, $(-2, -\frac{1}{6})$, $(3, -6)$

In Exercises 31–46, sketch the graph of the equation. Identify any intercepts and test for symmetry.

31. $y = x$
32. $y = x - 2$
33. $y = x + 3$
34. $y = 2x - 3$
35. $y = -3x + 2$
36. $y = -\dfrac{1}{2}x + 2$
37. $y = \dfrac{1}{2}x - 4$
38. $y = x^2 + 3$
39. $y = 1 - x^2$
40. $y = 2x^2 + x$
41. $y = x^3 + 2$
42. $y = \sqrt{9 - x^2}$
43. $y = (x + 2)^2$
44. $x = y^2 - 4$
45. $y = \dfrac{1}{x}$
46. $y = 2x^4$

C In Exercises 47–52, use a graphing utility to graph the equation. (It may be necessary to solve for y and plot two equations.) Identify any intercepts and test for symmetry.

47. $y = -2x^2 + x + 1$
48. $y = x^3 - 1$
49. $y = x\sqrt{25 - x^2}$
50. $y = \dfrac{5}{x^2 + 1} - 1$
51. $x^2 + 4y^2 = 4$
52. $9x^2 + y^2 = 9$

In Exercises 53–56, create an equation whose graph has the indicated property. (There is more than one correct answer.)

53. The graph has intercepts at $x = -2$, $x = 4$, and $x = 6$.
54. The graph has intercepts at $x = -\frac{5}{2}$, $x = 2$, and $x = \frac{3}{2}$.
55. The graph is symmetric with respect to the origin.
56. The graph is symmetric with respect to the x-axis.

In Exercises 57–64, find the points of intersection of the graphs of the equations and check your results.

57. $x + y = 2$
$2x - y = 1$

58. $2x - 3y = 13$
$5x + 3y = 1$

59. $x + y = 7$
$3x - 2y = 11$

60. $x^2 + y^2 = 25$
$2x + y = 10$

61. $x^2 + y^2 = 5$
$x - y = 1$

62. $x^2 + y = 4$
$2x - y = 1$

63. $y = x^3$
$y = x$

64. $x = 3 - y^2$
$y = x - 1$

SECTION 4 Lines in the Plane

The Slope of a Line • Equations of Lines • Sketching the Graph of a Line • Parallel and Perpendicular Lines

The Slope of a Line

The **slope** of a nonvertical line is a measure of the number of units a line rises (or falls) vertically for each unit of horizontal change from left to right. Consider the two points (x_1, y_1) and (x_2, y_2) on the line in Figure 36. As you move from left to right along this line, a vertical change of

$$\Delta y = y_2 - y_1 \qquad \text{Change in } y$$

units corresponds to a horizontal change of

$$\Delta x = x_2 - x_1 \qquad \text{Change in } x$$

units. (Δ is the Greek uppercase letter *delta,* and the symbols Δy and Δx are read "delta y" and "delta x.")

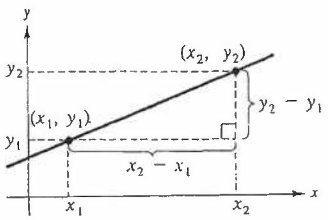


FIGURE 36
$\Delta y = y_2 - y_1 =$ change in y
$\Delta x = x_2 - x_1 =$ change in x

Definition of the Slope of a Line

The **slope** m of a nonvertical line passing through the points (x_1, y_1) and (x_2, y_2) is

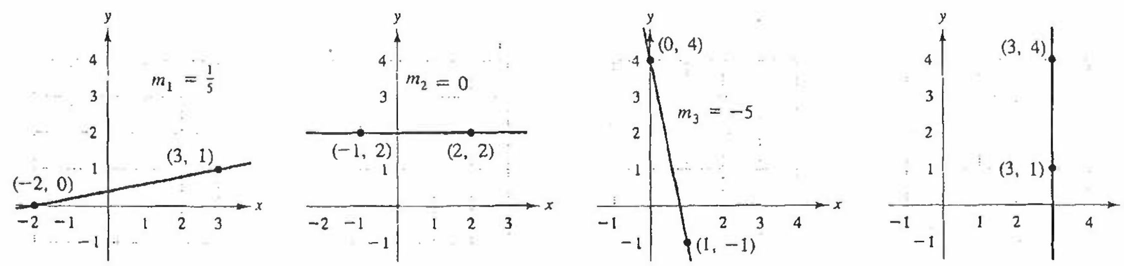
$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \qquad x_1 \neq x_2.$$

REMARK When using the formula for slope, note that

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-(y_1 - y_2)}{-(x_1 - x_2)} = \frac{y_1 - y_2}{x_1 - x_2}.$$

It does not matter in which order you subtract *as long as* you are consistent and both "subtracted coordinates" come from the same point.

Figure 37 shows four lines: one has a positive slope, one has a slope of zero, one has a negative slope, and one has an "undefined slope."



If m is positive, then the line rises.

If m is zero, then the line is horizontal.

If m is negative, then the line falls.

If the line is vertical, then the slope is undefined.

FIGURE 37

Equations of Lines

Any two points on a nonvertical line can be used to calculate its slope. This can be verified from the similar triangles shown in Figure 38. (Recall that the ratios of corresponding sides of similar triangles are equal.)

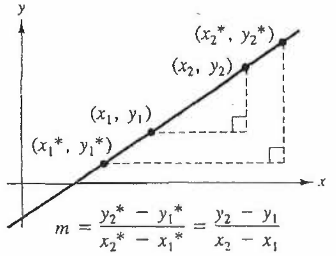


FIGURE 38
Any two points on a line can be used to determine its slope.

You can write an equation of a line if you know the slope of the line and the coordinates of one point on the line. Suppose the slope is m and the point is (x_1, y_1). If (x, y) is any other point on the line, then

$$\frac{y - y_1}{x - x_1} = m.$$

This equation, involving the two variables x and y, can be rewritten in the form $y - y_1 = m(x - x_1)$, which is called the **point-slope equation of a line.**

Point-Slope Equation of a Line

An equation of the line with slope m passing through the point (x_1, y_1) is given by

$$y - y_1 = m(x - x_1).$$

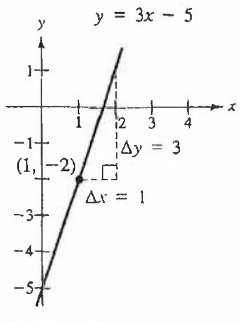


FIGURE 39
Point-slope equation of a line.

EXAMPLE 1 Finding an Equation of a Line

Find an equation of the line that has a slope of 3 and passes through the point $(1, -2)$.

Solution

$y - y_1 = m(x - x_1)$	Point-slope form
$y - (-2) = 3(x - 1)$	Substitute -2 for y_1, 1 for x_1, and 3 for m
$y + 2 = 3x - 3$	Simplify
$y = 3x - 5$	Solve for y

(See Figure 39.)

The slope of a line can be interpreted as either a *ratio* or a *rate.* If the x-axis and y-axis have the same unit of measure, then the slope has no units and is a **ratio.** If the x-axis and y-axis have different units of measure, then the slope is a rate or **rate of change.** As you study this text, you will encounter applications involving both interpretations of slope.

EXAMPLE 2 Population Growth and Engineering Design

a. The population of Arizona was 1,775,000 in 1970 and 2,718,000 in 1980. Over this 10-year period, the average rate of change of the population was

$$\text{Rate of change} = \frac{\text{Change in population}}{\text{Change in years}}$$
$$= \frac{2{,}718{,}000 - 1{,}775{,}000}{1980 - 1970}$$
$$= 94{,}300 \text{ people per year.}$$

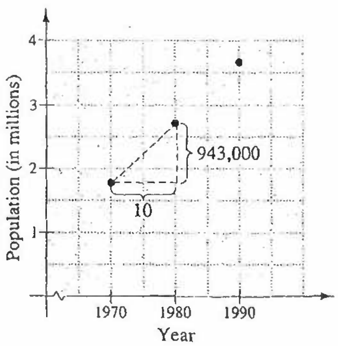


FIGURE 40

If Arizona's population had continued to increase at this same rate for the next 10 years, it would have had a 1990 population of 3,661,000. In the 1990 census, however, Arizona's population was determined to be 3,665,000, so the population's rate of change from 1980 to 1990 was a little greater than in the previous decade (see Figure 40).

b. In tournament water-ski jumping, the ramp rises to a height of 6 feet on a raft that is 21 feet long, as shown in Figure 41. The slope of the ski ramp is the ratio of its height (the rise) to the length of its base (the run).

$$\text{Slope of ramp} = \frac{\text{Rise}}{\text{Run}} = \frac{6 \text{ feet}}{21 \text{ feet}} = \frac{2}{7}$$

In this case, note that the slope is a ratio and has no units.

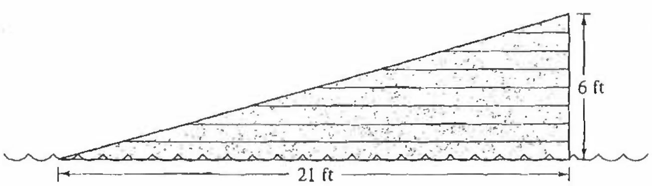


FIGURE 41
Dimensions of a water-ski ramp.

REMARK The rate of change found in Example 2a is an **average rate of change.** An average rate of change is always calculated over an interval. In this case, the interval is [1970, 1980]. Later in the text you will study another type of rate of change called an *instantaneous rate of change.*

Sketching the Graph of a Line

In Section 2, we mentioned that many problems in analytic geometry can be classified in two basic categories: (1) Given a graph, what is its equation? and (2) Given an equation, what is its graph? The point-slope equation of a line can be used to solve problems in the first category. However, this form is not especially useful for solving problems in the second category. The form that is better suited to sketching the graph of a line is the **slope-intercept** form for the equation of a line.

The Slope-Intercept Equation of a Line

The graph of the linear equation

$$y = mx + b$$

is a line having a *slope* of m and a *y-intercept* at $(0, b)$.

EXAMPLE 3 Sketching Lines in the Plane

Sketch the graphs of the equations.

a. $y = 2x + 1$ b. $y = 2$ c. $3y + x - 6 = 0$

Solution

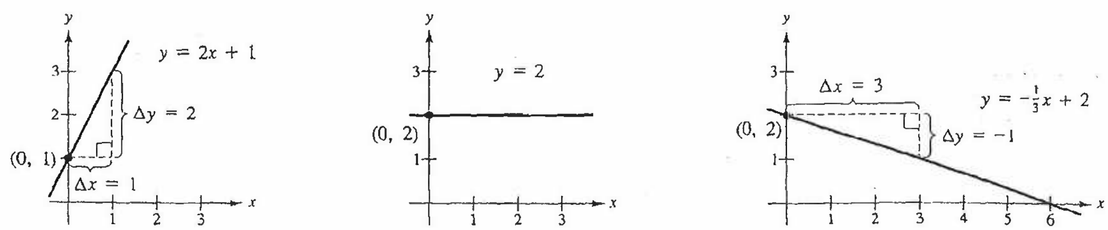
a. Because $b = 1$, the y-intercept is $(0, 1)$. Because the slope is $m = 2$, you know that the line rises 2 units for each unit it moves to the right, as shown in Figure 42(a).

b. Because $b = 2$, the y-intercept is $(0, 2)$. Because the slope is $m = 0$, you know that the line is horizontal, as shown in Figure 42(b).

c. Begin by writing the equation in slope-intercept form.

$$3y + x - 6 = 0 \quad \text{Original equation}$$
$$3y = -x + 6 \quad \text{Isolate } y\text{-term on the left}$$
$$y = -\frac{1}{3}x + 2 \quad \text{Slope-intercept form}$$

In this form, you can see that the y-intercept is $(0, 2)$ and the slope is $m = -\frac{1}{3}$. This means that the line falls 1 unit for every 3 units it moves to the right, as shown in Figure 42(c).



(a) $m = 2$; line rises. (b) $m = 0$; line is horizontal. (c) $m = -\frac{1}{3}$; line falls.

FIGURE 42

Because the slope of a vertical line is not defined, its equation cannot be written in the slope-intercept form. However, the equation of *any* line can written in the **general form**

$$Ax + By + C = 0$$

where A and B are not *both* zero. For instance, the vertical line given by $x = a$ can be represented by the general form $x - a = 0$.

Summary of Equations of Lines

1. General form: $Ax + By + C = 0$
2. Vertical line: $x = a$
3. Horizontal line: $y = b$
4. Point-slope form: $y - y_1 = m(x - x_1)$
5. Slope-intercept form: $y = mx + b$

REMARK Three points are **collinear** if they lie on the same line.

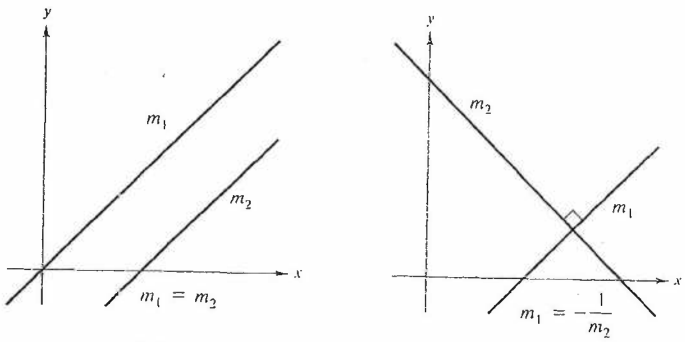
Parallel and Perpendicular Lines

The slope of a line is a convenient tool for determining whether two lines are parallel or perpendicular, as shown in Figure 43.

Parallel and Perpendicular Lines

1. Two distinct nonvertical lines are **parallel** if and only if their slopes are equal.
2. Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other. That is, if and only if

$$m_1 = -\frac{1}{m_2}.$$



Parallel Lines

Perpendicular Lines

FIGURE 43

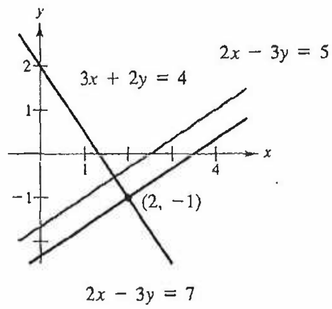


FIGURE 44
Lines parallel and perpendicular to $2x - 3y = 5$.

EXAMPLE 4 Finding Parallel and Perpendicular Lines

Find the general forms of the equations of the lines that pass through the point $(2, -1)$ and are

a. parallel to the line $2x - 3y = 5$
b. perpendicular to the line $2x - 3y = 5$

as shown in Figure 44.

Solution By writing the linear equation $2x - 3y = 5$ in slope-intercept form $y = \frac{2}{3}x - \frac{5}{3}$, you can see that the given line has a slope of $m = \frac{2}{3}$.

a. The line through $(2, -1)$ that is parallel to the given line also has a slope of $\frac{2}{3}$.

$$y - (-1) = \frac{2}{3}(x - 2) \qquad \text{Point-slope form}$$
$$3(y + 1) = 2(x - 2) \qquad \text{Simplify}$$
$$2x - 3y - 7 = 0 \qquad \text{General form}$$

Note the similarity to the original equation.

b. Using the negative reciprocal of the slope of the given line, you can determine that the slope of a line perpendicular to the given line is $-\frac{3}{2}$. Therefore, the line through the point $(2, -1)$ that is perpendicular to the given line has the following equation.

$$y - (-1) = -\frac{3}{2}(x - 2) \qquad \text{Point-slope form}$$
$$2(y + 1) = -3(x - 2) \qquad \text{Simplify}$$
$$3x + 2y - 4 = 0 \qquad \text{General form}$$

TECHNOLOGY The apparent slope of a line will be distorted if you use different tic-spacing on the x- and y-axes. For instance, the graphing calculator screens in Figures 45 and 46 both show the lines given by $y = 2x$ and $y = -\frac{1}{2}x + 3$. Because these lines have slopes that are negative reciprocals, they must be perpendicular. In Figure 45, however, the lines don't appear to be perpendicular because the tic-spacing on the x-axis is not the same as the tic-spacing on the y-axis.

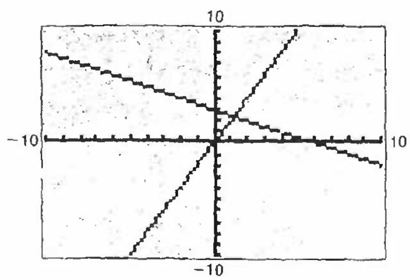


FIGURE 45
Tic-spacing on the x-axis is not the same as the tic-spacing on the y-axis.

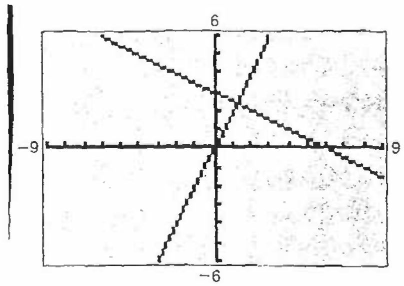


FIGURE 46
Tic-spacing on the x-axis is the same as the tic-spacing on the y-axis.

EXERCISES for Section 4

In Exercises 9–12, plot the pair of points and find the slope of the line passing through them.

9. $(3, -4), (5, 2)$ 10. $(2, 1), (2, 5)$
11. $(1, 2), (-2, 4)$ 12. $(\frac{7}{8}, \frac{3}{4}), (\frac{5}{4}, -\frac{1}{4})$

In Exercises 13–16, use the point on the line and the slope of the line to find three additional points that the line passes through. (There is more than one correct answer.)

	Point	*Slope*		*Point*	*Slope*
13.	$(2, 1)$	$m = 0$	14.	$(-3, 4)$	m undefined
15.	$(1, 7)$	$m = -3$	16.	$(-2, -2)$	$m = 2$

In Exercises 19–22, find the slope and the y-intercept (if possible) of the line.

19. $x + 5y = 20$ 20. $5x - 5y = 15$
21. $x = 4$ 22. $y = -1$

In Exercises 23–28, find an equation for the line that passes through the points, and sketch the line.

23. $(2, 1), (0, -3)$ 24. $(-3, -4), (1, 4)$
25. $(0, 0), (-1, 3)$ 26. $(-3, 6), (1, 2)$
27. $(1, -2), (3, -2)$ 28. $(\frac{7}{8}, \frac{3}{4}), (\frac{5}{4}, -\frac{1}{4})$

In Exercises 29–34, find an equation of the line that passes through the point and has the indicated slope. Sketch the line.

	Point	*Slope*		*Point*	*Slope*
29.	$(0, 3)$	$m = \frac{3}{4}$	30.	$(-1, 2)$	m undefined
31.	$(0, 0)$	$m = \frac{2}{3}$	32.	$(-2, 4)$	$m = -\frac{3}{5}$
33.	$(0, 2)$	$m = 4$	34.	$(0, 4)$	$m = 0$

35. Find an equation of the vertical line with x-intercept at 3.

36. Show that the line with intercepts $(a, 0)$ and $(0, b)$ has the following equation.

$$\frac{x}{a} + \frac{y}{b} = 1, \quad a \neq 0, b \neq 0$$

In Exercises 37–40, use the result of Exercise 36 to write an equation of the indicated line.

37. x-intercept: $(2, 0)$
y-intercept: $(0, 3)$

38. x-intercept $(-\frac{2}{3}, 0)$
y-intercept: $(0, -2)$

39. Point on line: $(1, 2)$
x-intercept: $(a, 0)$
y-intercept: $(0, a)$
$(a \neq 0)$

40. Point on line: $(-3, 4)$
x-intercept: $(a, 0)$
y-intercept: $(0, a)$
$(a \neq 0)$

In Exercises 41–46, write equations of the lines through the point (a) parallel to the given line and (b) perpendicular to the given line.

	Point	*Given Line*
41.	$(2, 1)$	$4x - 2y = 3$
42.	$(-3, 2)$	$x + y = 7$
43.	$(\frac{7}{8}, \frac{3}{4})$	$5x + 3y = 0$
44.	$(-6, 4)$	$3x + 4y = 7$
45.	$(2, 5)$	$x = 4$
46.	$(-1, 0)$	$y = -3$

In Exercises 55 and 56, determine whether the three points are collinear.

55. $(-2, 1), (-1, 0), (2, -2)$
56. $(0, 4), (7, -6), (-5, 11)$

Functions and Function Notation

A **relation** between two sets X and Y is a set of ordered pairs, each of the form (x, y) where x is a member of X and y is a member of Y. A **function** from X to Y is a relation between X and Y that has the property that any two ordered pairs with the same x-value also have the same y-value. The variable x is the **independent variable,** and the variable y is the **dependent variable.**

Many real-life situations can be modeled by functions. For instance, the area A of a circle is a function of the circle's radius r. In this case r is the independent variable and A is the dependent variable.

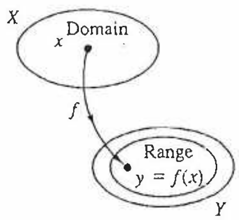


FIGURE 47
A real-valued function f of a real variable.

Definition of a Real-Valued Function of a Real Variable

Let X and Y be sets of real numbers. A **real-valued function f of a real variable** x from X to Y is a correspondence that assigns to each number x in X exactly one number y in Y.

The **domain** of f is the set X. The number y is the **image** of x under f and is denoted by $f(x)$. The **range** of f is a subset of Y and consists of all images of numbers in X (see Figure 47).

Functions can be specified in a variety of ways. In this text, however, we will concentrate primarily on functions that are given by equations involving the dependent and independent variables. For instance, the equation $x + 2y = 1$ defines y, the dependent variable, as a function of x, the independent variable. To **evaluate** this function (that is, to find the y-value that corresponds to a given x-value), it is convenient to isolate y on the left side of the equation.

$$y = \frac{1}{2}(1 - x)$$

Using f as the name of the function, you can write this equation as

$$f(x) = \frac{1}{2}(1 - x). \qquad \text{Function notation}$$

Function notation has the advantage of clearly identifying the dependent variable as $f(x)$ while at the same time telling you that x is the independent variable and that the function itself is "f." The notation $f(x)$ is read "f of x." Function notation allows you to be less wordy. Instead of asking "What is the value of y that corresponds to $x = 3$?" you can ask "What is $f(3)$?"

In an equation that defines a function, the role of the variable x is simply that of a placeholder. For instance, the function given by

$$f(x) = 2x^2 - 4x + 1$$

can be described by the form

$$f(\ \ ) = 2(\ \ )^2 - 4(\ \ ) + 1$$

where parentheses are used instead of x. To evaluate $f(-2)$, simply place -2 in each set of parentheses.

$$\begin{aligned} f(-2) &= 2(-2)^2 - 4(-2) + 1 \\ &= 2(4) + 8 + 1 \\ &= 17 \end{aligned}$$

REMARK Although f is often used as a convenient function name and x as the independent variable, you can use other symbols. For instance, the following equations all define the same function.

$$f(x) = x^2 - 4x + 7, \qquad f(t) = t^2 - 4t + 7, \qquad g(s) = s^2 - 4s + 7$$

EXAMPLE 1 Evaluating a Function

For the function f defined by $f(x) = x^2 + 7$, evaluate the following.

a. $f(3a)$ b. $f(b - 1)$ c. $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}, \quad \Delta x \neq 0$

Solution

a. $f(3a) = (3a)^2 + 7$ Replace x with $3a$

$= 9a^2 + 7$ Simplify

b. $f(b - 1) = (b - 1)^2 + 7$ Replace x with $b - 1$

$= b^2 - 2b + 1 + 7$ Expand binomial

$= b^2 - 2b + 8$ Simplify

c.
$$\begin{aligned}\frac{f(x + \Delta x) - f(x)}{\Delta x} &= \frac{[(x + \Delta x)^2 + 7] - (x^2 + 7)}{\Delta x}\\ &= \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 7 - x^2 - 7}{\Delta x}\\ &= \frac{2x\Delta x + (\Delta x)^2}{\Delta x}\\ &= \frac{\Delta x\,(2x + \Delta x)}{\Delta x}\\ &= 2x + \Delta x, \qquad \Delta x \neq 0\end{aligned}$$

REMARK The expression in Example 1c is called a difference quotient and has special significance in calculus. We will say more about this in Chapter 2.

The Domain and Range of a Function

The domain of a function can be described explicitly, or it may be described *implicitly* by an equation used to define the function. The implied domain is the set of all real numbers for which the equation is defined. For example, the function given by

$$f(x) = \frac{1}{x^2 - 4}, \qquad 4 \le x \le 5$$

has an explicitly defined domain given by $\{x: 4 \le x \le 5\}$. On the other hand, the function given by

$$g(x) = \frac{1}{x^2 - 4}$$

has an implied domain which is the set $\{x: x \neq \pm 2\}$.

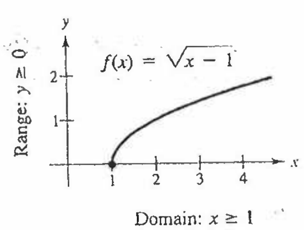


FIGURE 48
The domain of $f(x)$ is $[1, \infty)$ and ' .e range is $[0, \infty)$.

EXAMPLE 2 Finding the Domain and Range of a Function

a. The domain of the function

$$f(x) = \sqrt{x - 1}$$

is the set of all x-values for which $x - 1 \ge 0$, which is the interval $[1, \infty)$. To find the range, observe that $f(x) = \sqrt{x - 1}$ is never negative. Moreover, as x takes on the various values in the domain, $f(x)$ takes on all nonnegative values. Thus, the range is the interval $[0, \infty)$, as indicated in Figure 48.

b. The domain of the function

$$f(x) = \sqrt{x^2 - x - 6}$$

is the set of all x-values such that $x^2 - x - 6 \geq 0$. Using the techniques shown in Example 4 in Section 1, you can conclude that the domain is

$$(-\infty, -2] \cup [3, \infty).$$

EXAMPLE 3 A Function Defined by More than One Equation

Determine the domain and range of the following function.

$$f(x) = \begin{cases} 1 - x, & \text{if } x < 1 \\ \sqrt{x - 1}, & \text{if } x \geq 1 \end{cases}$$

Solution Because f is defined for $x < 1$ and $x \geq 1$, the domain is the entire set of real numbers. On the portion of the domain for which $x \geq 1$, the function behaves as in Example 2. For $x < 1$, the values of $1 - x$ are positive. Therefore, the range of the function is the interval

$[0, \infty)$. Range

(See Figure 49.)

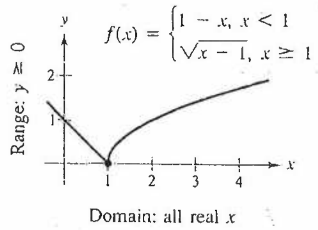


FIGURE 49
The domain of $f(x)$ is $(-\infty, \infty)$ and the range is $[0, \infty)$.

A function from X to Y is **one-to-one** if to each y-value in the range there corresponds exactly one x-value in the domain. For instance, the function given in Example 2a is one-to-one, whereas the function given in Example 3 is not one-to-one. If the range consists of all of Y, then the function is called **onto.**

The Graph of a Function

The graph of the function $y = f(x)$ consists of all points $(x, f(x))$, where x is in the domain of f. In Figure 50, note that

$$x = \text{the directed distance from the } y\text{-axis}$$
$$f(x) = \text{the directed distance from the } x\text{-axis}.$$

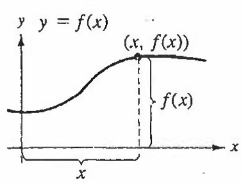
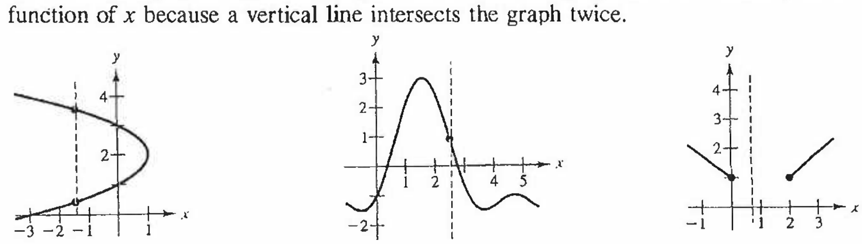


FIGURE 50
The graph of a function.

A vertical line can intersect the graph of a function of x at most *once.* This observation provides a convenient visual test (called the **vertical line test**) for functions of x. For example, in Figure 51(a), you can see that the graph does not define y as a function of x because a vertical line intersects the graph twice.

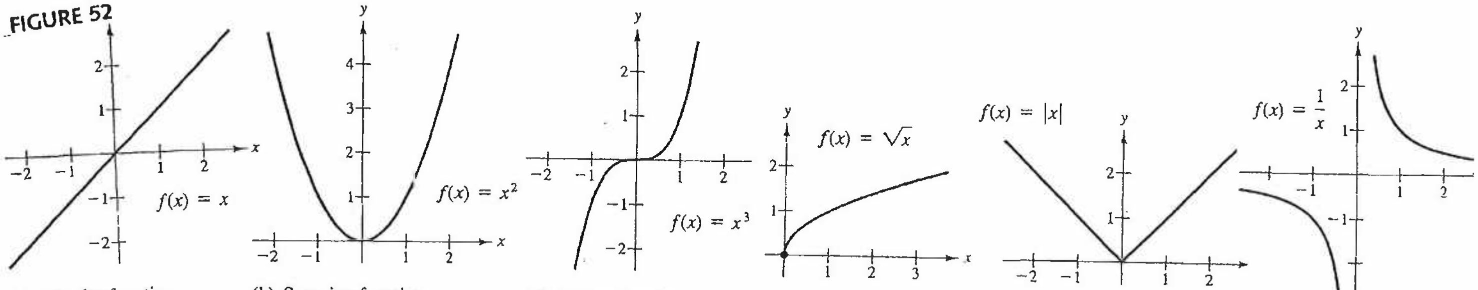


(a) Not a function of x (b) A function of x (c) A function of x

FIGURE 51
The vertical line test for functions.

Figure 52 shows the graphs of six basic functions. You should be able to recognize these graphs.

FIGURE 52



(a) Identity function (b) Squaring function (c) Cubing function (d) Square root function (e) Absolute value function (f) Rational function

Transformations of Functions

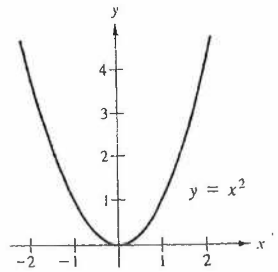


FIGURE 53
The original graph.

Some families of graphs have the same basic shape. For example, consider the graph of $y = x^2$, as shown in Figure 53. Now compare this graph with those shown in Figure 54.

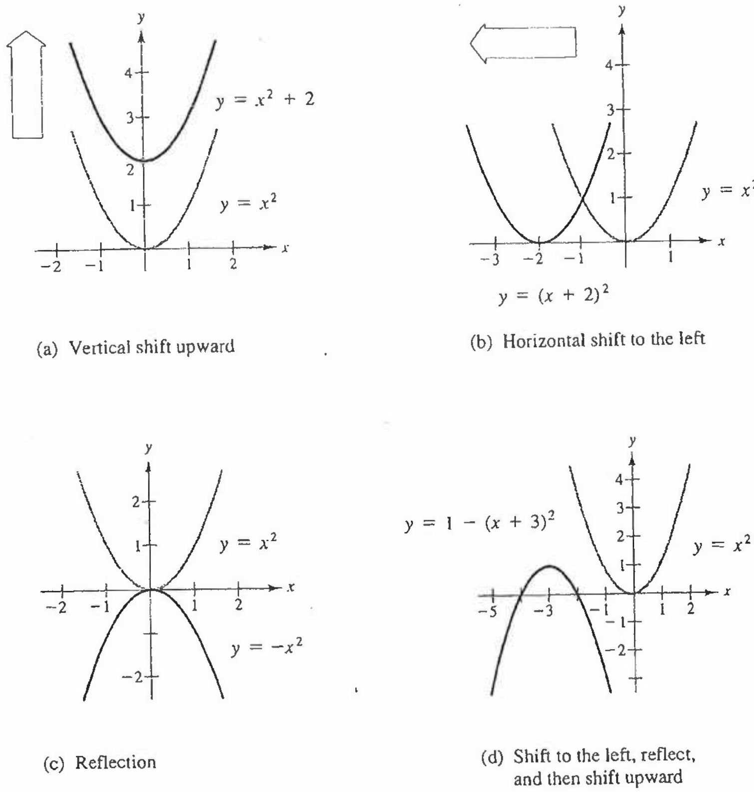


FIGURE 54
Transformations of the graph of $y = x^2$.

Each of the graphs in Figure 54 is a **transformation** of the graph of $y = x^2$. The three basic types of transformations illustrated by these graphs are vertical shifts, horizontal shifts, and reflections. Function notation lends itself well to describing transformations of graphs in the plane.

Basic Types of Transformations ($c > 0$)

Original graph:	$y = f(x)$
Horizontal shift c units to the **right:**	$y = f(x - c)$
Horizontal shift c units to the **left:**	$y = f(x + c)$
Vertical shift c units **downward:**	$y = f(x) - c$
Vertical shift c units **upward:**	$y = f(x) + c$
Reflection (about the x-axis):	$y = -f(x)$
Reflection (about the y-axis):	$y = f(-x)$

Classifications and Combinations of Functions

The modern notion of a function was derived from the efforts of many seventeenth- and eighteenth-century mathematicians. Of particular note was Leonhard Euler (1707–1783), to whom we are indebted for the function notation $y = f(x)$. By the end of the eighteenth century, mathematicians and scientists had concluded that most real-world phenomena could be represented by mathematical models taken from a basic collection of functions called **elementary functions.**

Elementary functions are divided into three categories: (1) algebraic, (2) trigonometric, and (3) logarithmic and exponential. We will review the trigonometric functions in Section 6 of this preliminary chapter and introduce the remaining elementary functions in Chapter 5.

The most common type of algebraic function is a **polynomial function**

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0, \qquad a_n \neq 0$$

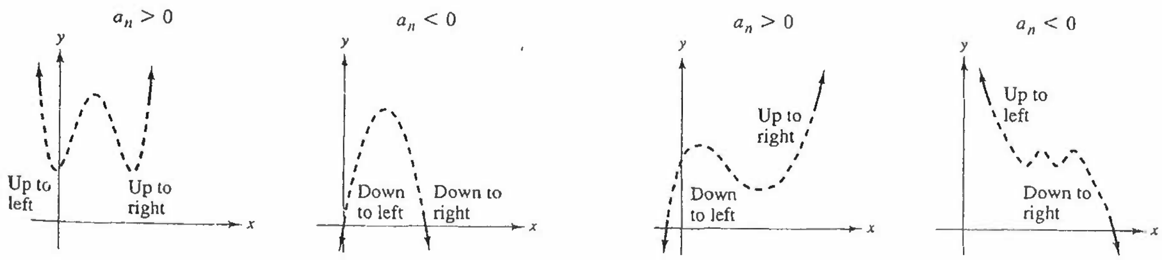
where the positive integer n is the **degree** of the polynomial function. The numbers a_i are **coefficients,** with a_n the **leading coefficient** and a_0 the **constant term** of the polynomial function. It is common practice to use subscript notation for coefficients of general polynomial functions, but for polynomial functions of low degree, the following simpler forms are often used.

Zeroth degree:	$f(x) = a$	Constant function
First degree:	$f(x) = ax + b$	Linear function
Second degree:	$f(x) = ax^2 + bx + c$	Quadratic function
Third degree:	$f(x) = ax^3 + bx^2 + cx + d$	Cubic function

Although the graph of a polynomial function can have several turns, eventually the graph will rise or fall without bound as x moves to the right or left. Whether the graph of

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

eventually rises or falls can be determined by the function's degree (odd or even) and by the leading coefficient a_n, as indicated in Figure 55. Note that the dashed portions of the graphs indicate that the **leading coefficient test** determines *only* the right and left behavior of the graph.



Graphs of polynomial functions of even degree

Graphs of polynomial functions of odd degree

FIGURE 55
The leading coefficient test for polynomial functions.

Just as a rational number can be written as the quotient of two integers, a **rational function** can be written as the quotient of two polynomials. Specifically, a function f is rational if it has the form

$$f(x) = \frac{p(x)}{q(x)}, \qquad q(x) \neq 0$$

where $p(x)$ and $q(x)$ are polynomials.

Polynomial functions and rational functions are examples of **algebraic functions.** An algebraic function of x is one that can be expressed as a finite number of sums, differences, multiples, quotients, and radicals involving x^n. For example, $f(x) = \sqrt{x + 1}$ is algebraic. Functions that are not algebraic are **transcendental.** For instance, the trigonometric functions discussed in Section 6 are transcendental.

Two functions can be combined in various ways to create new functions. For example, given $f(x) = 2x - 3$ and $g(x) = x^2 + 1$, you can form the following functions.

$f(x) + g(x) = (2x - 3) + (x^2 + 1) = x^2 + 2x - 2$	Sum
$f(x) - g(x) = (2x - 3) - (x^2 + 1) = -x^2 + 2x - 4$	Difference
$f(x)g(x) = (2x - 3)(x^2 + 1) = 2x^3 - 3x^2 + 2x - 3$	Product
$\dfrac{f(x)}{g(x)} = \dfrac{2x - 3}{x^2 + 1}$	Quotient

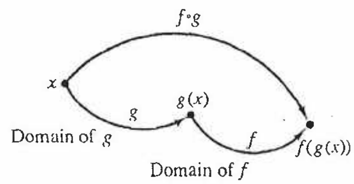


FIGURE 56
The domain of the composite function $f \circ g$.

You can combine two functions in yet another way to form a **composite function.**

Definition of Composite Function

Let f and g be functions. The function given by $(f \circ g)(x) = f(g(x))$ is called the **composite** of f with g. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f (see Figure 56).

The composite of f with g may not be equal to the composite of g with f.

EXAMPLE 4 Finding Composites of Functions

Given $f(x) = 2x - 3$ and $g(x) = x^2 + 1$, find

a. $f \circ g$ **b.** $g \circ f$.

Solution

a. $(f \circ g)(x) = f(g(x)) = 2(g(x)) - 3 = 2(x^2 + 1) - 3$
$= 2x^2 - 1$

b. $(g \circ f)(x) = g(f(x)) = (f(x))^2 + 1 = (2x - 3)^2 + 1$
$= 4x^2 - 12x + 10$

Note that $(f \circ g)(x) \neq (g \circ f)(x)$.

In Section 3, we defined an x-intercept of a graph to be a point $(a, 0)$ at which the graph crosses the x-axis. If the graph represents a function f, then the number a is a **zero** of f. In other words, the zeros of a function f are the solutions of the equation $f(x) = 0$. For example, the function $f(x) = x - 4$ has a zero at $x = 4$ because $f(4) = 0$.

In Section 3 we also discussed different types of symmetry. In the terminology of functions, a function is **even** if its graph is symmetric with respect to the y-axis, and is **odd** if its graph is symmetric with respect to the origin. The symmetry tests in Section 3 yield the following test for even and odd functions.

Test for Even and Odd Functions

The function $y = f(x)$ is **even** if $f(-x) = f(x)$.
The function $y = f(x)$ is **odd** if $f(-x) = -f(x)$.

REMARK Except for the constant function $f(x) = 0$, the graph of a function cannot have symmetry with respect to the x-axis because it then would fail the vertical line test for the graph of a function.

EXAMPLE 5 Even and Odd Functions and Zeros of Functions

Determine whether the following functions are even, odd, or neither. Then find the zeros of the functions.

a. $f(x) = x^3 - x$ **b.** $g(x) = x^2 + 1$

Solution

a. This function is odd because

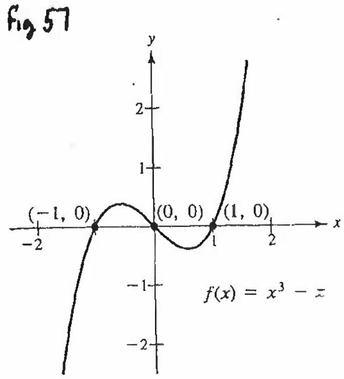
$$f(-x) = (-x)^3 - (-x)$$
$$= -x^3 + x$$
$$= -(x^3 - x)$$
$$= -f(x).$$

The zeros of f are found as follows.

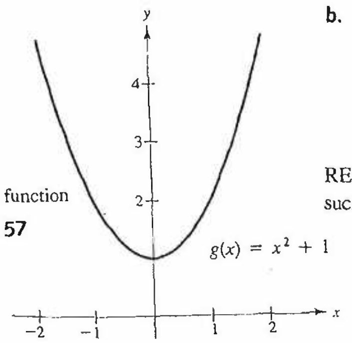
$x^3 - x = 0$	Let $f(x) = 0$
$x(x^2 - 1) = x(x - 1)(x + 1) = 0$	Factor
$x = 0, 1, -1$	Zeros of f

See Figure 57(a).

Fig 57



(a) Odd function



(b) Even function

FIGURE 57

b. This function is even because

$$g(-x) = (-x)^2 + 1 = x^2 + 1 = g(x).$$

It has no zeros because $x^2 + 1$ is positive for all x. See Figure 57(b).

REMARK Each of the functions in Example 5 is either even or odd. However, some functions, such as $f(x) = x^2 + x + 1$, are neither even nor odd.

EXERCISES for Section 5

In Exercises 1–12, evaluate (if possible) the function at the specified value(s) of the independent variable and simplify the results.

1. $f(x) = 2x - 3$
   a. $f(0)$
   b. $f(-3)$
   c. $f(b)$
   d. $f(x - 1)$
2. $f(x) = x^2 - 2x + 2$
   a. $f(\frac{1}{2})$
   b. $f(-1)$
   c. $f(c)$
   d. $f(x + \Delta x)$
3. $f(x) = \sqrt{x + 3}$
   a. $f(-2)$
   b. $f(6)$
   c. $f(c)$
   d. $f(x + \Delta x)$
4. $f(x) = |x| + 4$
   a. $f(2)$
   b. $f(-2)$
   c. $f(x^2)$
   d. $f(x + \Delta x) - f(x)$
5. $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$
   a. $f(-1)$ b. $f(0)$ c. $f(2)$ d. $f(t^2 + 1)$
6. $f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ 2x^2 + 2, & x > 1 \end{cases}$
   a. $f(-2)$ b. $f(0)$ c. $f(1)$ d. $f(s^2 + 2)$
7. $f(x) = x^2 - x + 1$
   $\dfrac{f(2 + \Delta x) - f(2)}{\Delta x}$
8. $f(x) = \dfrac{1}{x}$
   $\dfrac{f(1 + \Delta x) - f(1)}{\Delta x}$
9. $f(x) = x^3$
   $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$
10. $f(x) = 3x - 1$
   $\dfrac{f(x) - f(1)}{x - 1}$
11. $f(x) = \dfrac{1}{\sqrt{x - 1}}$
   $\dfrac{f(x) - f(2)}{x - 2}$
12. $f(x) = x^3 - x$
   $\dfrac{f(x) - f(1)}{x - 1}$

In Exercises 13–20, find the domain and range of the function, and sketch its graph.

13. $f(x) = 4 - x$
14. $g(x) = \dfrac{4}{x}$
15. $h(x) = \sqrt{x - 1}$
16. $f(x) = \frac{1}{2}x^3 + 2$
17. $f(x) = \sqrt{9 - x^2}$
18. $h(x) = \sqrt{25 - x^2}$
19. $f(x) = |x - 2|$
20. $f(x) = \dfrac{|x|}{x}$

In Exercises 29–36, determine whether y is a function of x.

29. $x^2 + y^2 = 4$
30. $x = y^2$
31. $x^2 + y = 4$
32. $x + y^2 = 4$
33. $2x + 3y = 4$
34. $x^2 + y^2 - 4y = 0$
35. $y^2 = x^2 - 1$
36. $x^2y - x^2 + 4y = 0$

In Exercises 41 and 42, specify a sequence of transformations that will yield the graph of each function from the graph of the function $f(x) = x^3$.

41. a. $g(x) = 4 - x^3$
    b. $g(x) = (x - 4)^3 + 2$
42. a. $h(x) = (x + 2)^3 + 1$
    b. $h(x) = 5 - (x - 1)^3$
43. Given $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$, find the composite functions.
    a. $f(g(1))$
    b. $g(f(1))$
    c. $g(f(0))$
    d. $f(g(-4))$
    e. $f(g(x))$
    f. $g(f(x))$
44. Given $f(x) = 1/x$ and $g(x) = x^2 - 1$, find the composite functions.
    a. $f(g(2))$
    b. $g(f(2))$
    c. $f\left(g\left(\dfrac{1}{\sqrt{2}}\right)\right)$
    d. $g\left(f\left(\dfrac{1}{\sqrt{2}}\right)\right)$
    e. $g(f(x))$
    f. $f(g(x))$

In Exercises 53–56, determine whether the function is even, odd, or neither.

53. $f(x) = 4 - x^2$
54. $f(x) = \sqrt[3]{x}$
55. $f(x) = x(4 - x^2)$
56. $f(x) = 4x - x^2$

67. *Area* A rectangle has a perimeter of 100 feet (see figure). Express the area A of the rectangle as a function of x.

68. *Area* You have 200 feet of fencing to enclose two adjacent rectangular fields (see figure). Express the area A of the enclosures as a function of x.

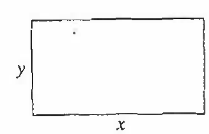


FIGURE FOR 67

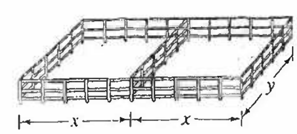


FIGURE FOR 68

69. *Volume* An open box is to be made from a square piece of material 12 inches on a side by cutting equal squares from each corner and turning up the sides (see figure). Express the volume V as a function of x.

70. *Area* A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{25 - x^2}$ (see figure). Write the area A of the rectangle as a function of x.

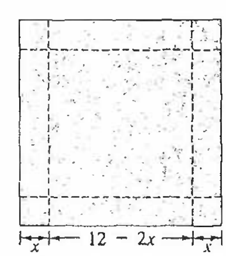


FIGURE FOR 69

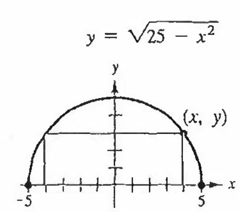


FIGURE FOR 70

Angles and Degree Measure • Radian Measure • The Trigonometric Functions • Evaluating Trigonometric Functions • Solving Trigonometric Equations • Graphs of Trigonometric Functions

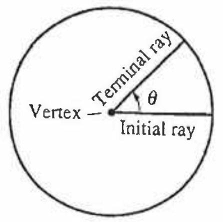


FIGURE 58
Standard position of an angle.

Angles and Degree Measure

An **angle** has three parts: an **initial ray,** a **terminal ray,** and a **vertex** (the point of intersection of the two rays), as shown in Figure 58. An angle is in **standard position** if its initial ray coincides with the positive x-axis and its vertex is at the origin. We assume that you are familiar with the degree measure of an angle.* It is common practice to use θ (the Greek lowercase letter *theta*) to represent both an angle and its measure. Angles between 0° and 90° are **acute** and angles between 90° and 180° are **obtuse.**

Positive angles are measured *counterclockwise,* and negative angles are measured *clockwise.* For instance, Figure 59 shows an angle whose measure is −45°. You cannot assign a measure to an angle by simply knowing where its initial and terminal rays are located. To measure an angle, you must also know how the terminal ray was revolved. For example, Figure 59 shows that the angle measuring −45° has the same terminal ray as the angle measuring 315°. Such angles are **coterminal.** In general, if θ is any angle, then

$$\theta + n(360), \qquad n \text{ is a nonzero integer}$$

is coterminal with θ.

An angle that is larger than 360° is one whose terminal ray has been revolved more than one full revolution counterclockwise, as shown in Figure 60. You can form an angle whose measure is less than −360° by revolving a terminal ray more than one full revolution clockwise.

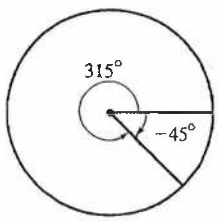


FIGURE 59
Coterminal angles

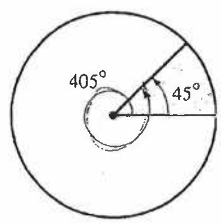


FIGURE 60
Coterminal angles

REMARK It is common to use the symbol θ to refer to both an *angle* and its *measure.* For instance, in Figure 60, you can write the measure of the smaller angle as $\theta = 45°$.

Radian Measure

To assign a radian measure to an angle θ, consider θ to be a central angle of a circle of radius 1, as shown in Figure 61. The **radian measure** of θ is then defined to be the length of the arc of the sector. Because the circumference of a circle is $2\pi r$, the circumference of a **unit circle** (of radius 1) is 2π. This implies that the radian measure of an angle measuring 360° is 2π. In other words, $360° = 2\pi$ radians.

Using radian measure for θ, the length s of a circular arc of radius r is $s = r\theta$, as shown in Figure 62.

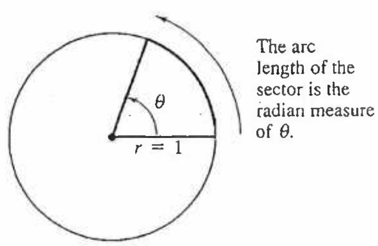


FIGURE 61
Unit circle

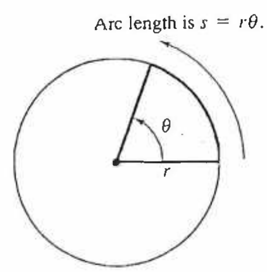


FIGURE 62
Circle of radius r

You should know the conversions of the common angles shown in Figure 63. For other angles, use the fact that 180° is equal to π radians.

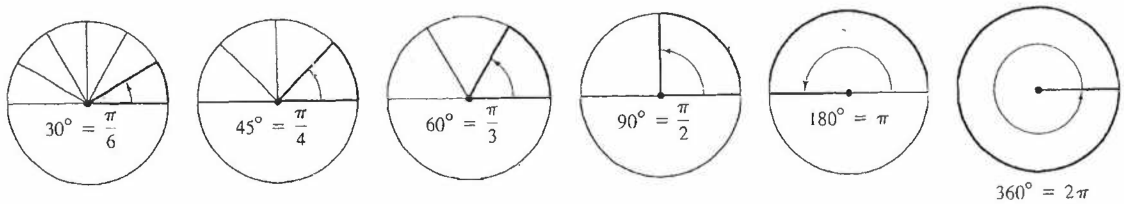


FIGURE 63
Radian and degree measure for several common angles.

EXAMPLE 1 Conversions Between Degrees and Radians

a. $40° = (40 \text{ deg})\left(\frac{\pi \text{ rad}}{180 \text{ deg}}\right) = \frac{2\pi}{9}$ radians

b. $-270° = (-270 \text{ deg})\left(\frac{\pi \text{ rad}}{180 \text{ deg}}\right) = -\frac{3\pi}{2}$ radians

c. $-\frac{\pi}{2}$ radians $= \left(-\frac{\pi}{2} \text{ rad}\right)\left(\frac{180 \text{ deg}}{\pi \text{ rad}}\right) = -90°$

d. $\frac{9\pi}{2}$ radians $= \left(\frac{9\pi}{2} \text{ rad}\right)\left(\frac{180 \text{ deg}}{\pi \text{ rad}}\right) = 810°$

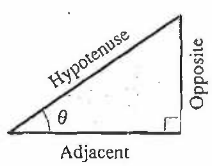


FIGURE 64
Sides of a right triangle.

The Trigonometric Functions

There are two common approaches to the study of trigonometry. In one, the trigonometric functions are defined as ratios of two sides of a right triangle. In the other, these functions are defined in terms of a point on the terminal side of an angle in standard position. We define the six trigonometric functions, **sine, cosine, tangent, cotangent, secant,** and **cosecant** (abbreviated as sin, cos, etc.), from both viewpoints.

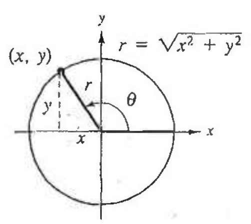


FIGURE 65
An angle in standard position.

Definition of the Six Trigonometric Functions

Right triangle definitions, where $0 < \theta < \frac{\pi}{2}$ (see Figure 64).

$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} \qquad \cos\theta = \frac{\text{adj.}}{\text{hyp.}} \qquad \tan\theta = \frac{\text{opp.}}{\text{adj.}}$$

$$\csc\theta = \frac{\text{hyp.}}{\text{opp.}} \qquad \sec\theta = \frac{\text{hyp.}}{\text{adj.}} \qquad \cot\theta = \frac{\text{adj.}}{\text{opp.}}$$

Circular function definitions, where θ is any angle (see Figure 65).

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$

The following trigonometric identities are direct consequences of the definitions. (ϕ is the Greek letter *phi.*)

Trigonometric Identities [*Note that* $\sin^2\theta$ *is used to represent* $(\sin\theta)^2$.]

Pythagorean identities:

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$\cot^2\theta + 1 = \csc^2\theta$

Reduction formulas:

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\tan(-\theta) = -\tan\theta$

$\sin\theta = -\sin(\theta - \pi)$

$\cos\theta = -\cos(\theta - \pi)$

$\tan\theta = \tan(\theta - \pi)$

Sum or difference of two angles:

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

$$\tan(\theta \pm \phi) = \frac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$$

Half-angle formulas:

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$

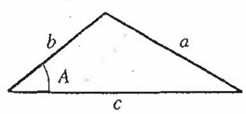
Double angle formulas:

$\sin 2\theta = 2\sin\theta\cos\theta$

$$\begin{aligned}\cos 2\theta &= 2\cos^2\theta - 1\\ &= 1 - 2\sin^2\theta\\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$

Law of Cosines:

$a^2 = b^2 + c^2 - 2bc\cos A$



Law of Cosines

Reciprocal identities:

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

Quotient identities:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

Evaluating Trigonometric Functions

There are two ways to evaluate trigonometric functions: (1) decimal approximations with a calculator (or a table of trigonometric values) and (2) exact evaluations using trigonometric identities and formulas from geometry. When using a calculator to evaluate a trigonometric function, remember to set the calculator to the appropriate mode—degree mode or radian mode.

EXAMPLE 2 Exact Evaluation of Trigonometric Functions

Evaluate the sine, cosine, and tangent of $\frac{\pi}{3}$.

Solution Begin by drawing the angle $\theta = \pi/3$ in the standard position, as shown in Figure 66. Then, because $60° = \pi/3$ radians, you can draw an equilateral triangle with sides of length 1 and θ as one of its angles. Because the altitude of this triangle bisects its base, you know that $x = \frac{1}{2}$. Using the Pythagorean Theorem, you obtain

$$y = \sqrt{r^2 - x^2} = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}.$$

Now, knowing the values of x, y, and r, you can write the following.

$$\sin\frac{\pi}{3} = \frac{y}{r} = \frac{\sqrt{3}/2}{1} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{3} = \frac{x}{r} = \frac{1/2}{1} = \frac{1}{2}$$

$$\tan\frac{\pi}{3} = \frac{y}{x} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

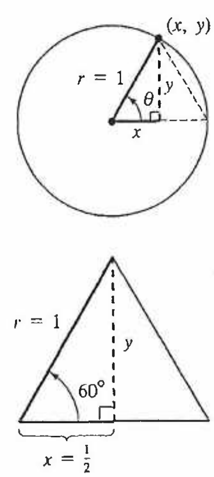


FIGURE 66
The angle $\pi/3$ in standard position.

REMARK All angles in the remainder of this text are measured in radians unless stated otherwise. For example, when we write $\sin 3$, we mean the sine of three radians, and when we write $\sin 3°$, we mean the sine of three degrees.

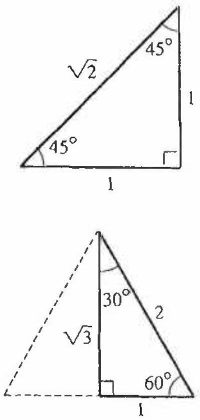


FIGURE 67
Common angles.

The degree and radian measures of several common angles are given in Table 4, along with the corresponding values of the sine, cosine, and tangent (see Figure 67).

TABLE 4
Common First Quadrant Angles

Degrees	0	30°	45°	60°	90°
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	Undefined

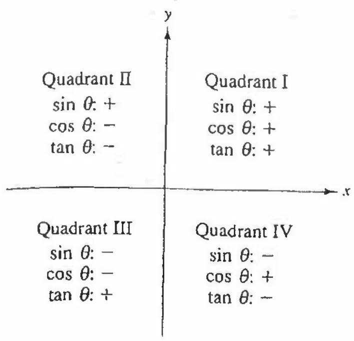


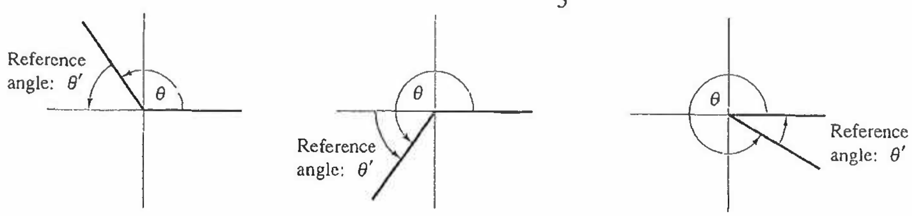
FIGURE 68
Quadrant signs for trigonometric functions.

The quadrant signs of the sine, cosine, and tangent functions are shown in Figure 68. To extend the use of Table 4 to angles in quadrants other than the first quadrant, you can use the concept of a **reference angle** (see Figure 69), with the appropriate quadrant sign. For instance, the reference angle for $3\pi/4$ is $\pi/4$, and because the sine is positive in the second quadrant, you can write

$$\sin\frac{3\pi}{4} = +\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

Similarly, because the reference angle for 330° is 30°, and the tangent is negative in the fourth quadrant, you can write

$$\tan 330° = -\tan 30° = -\frac{\sqrt{3}}{3}.$$



Quadrant II
$\theta' = \pi - \theta$ (radians)
$\theta' = 180° - \theta$ (degrees)

Quadrant III
$\theta' = \theta - \pi$ (radians)
$\theta' = \theta - 180°$ (degrees)

Quadrant IV
$\theta' = 2\pi - \theta$ (radians)
$\theta' = 360° - \theta$ (degrees)

FIGURE 69

EXAMPLE 3 Trigonometric Identities and Calculators

Evaluate the trigonometric expression.

a. $\sin\left(-\frac{\pi}{3}\right)$ **b.** $\sec 60°$ **c.** $\cos(1.2)$

Solution

a. Using the reduction formula $\sin(-\theta) = -\sin\theta$, you can write

$$\sin\left(-\frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}.$$

b. Using the reciprocal identity $\sec\theta = 1/\cos\theta$, you can write

$$\sec 60° = \frac{1}{\cos 60°} = \frac{1}{1/2} = 2.$$

c. Using a calculator, you can obtain

$$\cos(1.2) \approx 0.3624.$$

Remember that 1.2 is given in *radian* measure. Consequently, your calculator must be set in radian mode.

Solving Trigonometric Equations

How would you solve the equation $\sin\theta = 0$? You know that $\theta = 0$ is one solution, but this is not the only solution. Any one of the following values of θ is also a solution.

$\ldots, -3\pi, -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, \ldots$

You can write this infinite solution set as $\{n\pi : n \text{ is an integer}\}$.

EXAMPLE 4 Solving a Trigonometric Equation

Solve the equation

$$\sin\theta = -\frac{\sqrt{3}}{2}.$$

Solution To solve the equation, you should consider that the sine is negative in Quadrants III and IV and that

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

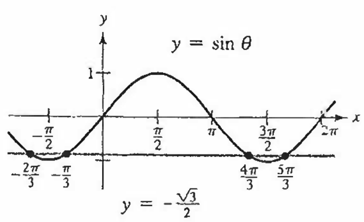


FIGURE 70

Solution points of $\sin\theta = \dfrac{-\sqrt{3}}{2}$.

Thus, you are seeking values of θ in the third and fourth quadrants that have a reference angle of $\pi/3$. In the interval $[0, 2\pi]$, the two angles fitting these criteria are

$$\theta = \pi + \frac{\pi}{3} = \frac{4\pi}{3} \quad \text{and} \quad \theta = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}.$$

By adding integer multiples of 2π to each of these solutions, you obtain the following general solution.

$$\theta = \frac{4\pi}{3} + 2n\pi \quad \text{or} \quad \theta = \frac{5\pi}{3} + 2n\pi, \qquad \text{where } n \text{ is an integer.}$$

(See Figure 70.)

EXAMPLE 5 Solving a Trigonometric Equation

Solve $\cos 2\theta = 2 - 3\sin\theta$, where $0 \le \theta \le 2\pi$.

Solution Using the double angle identity $\cos 2\theta = 1 - 2\sin^2\theta$, you can rewrite the equation as follows.

$$\begin{aligned} \cos 2\theta &= 2 - 3\sin\theta && \text{Given equation} \\ 1 - 2\sin^2\theta &= 2 - 3\sin\theta && \text{Trigonometric identity} \\ 0 &= 2\sin^2\theta - 3\sin\theta + 1 && \text{Quadratic form} \\ 0 &= (2\sin\theta - 1)(\sin\theta - 1) && \text{Factor} \end{aligned}$$

If $2\sin\theta - 1 = 0$, then $\sin\theta = 1/2$ and $\theta = \pi/6$ or $\theta = 5\pi/6$. If $\sin\theta - 1 = 0$, then $\sin\theta = 1$ and $\theta = \pi/2$. Thus, for $0 \le \theta \le 2\pi$, there are three solutions.

$$\theta = \frac{\pi}{6}, \quad \frac{5\pi}{6}, \quad \text{and} \quad \frac{\pi}{2}$$

Graphs of Trigonometric Functions

A function f is **periodic** if there exists a nonzero number p such that $f(x + p) = f(x)$ for all x in the domain of f. The smallest such positive value of p (if it exists) is the **period** of f. The sine, cosine, secant, and cosecant functions each have a period of 2π, and the other two trigonometric functions have a period of π, as shown in Figure 71.

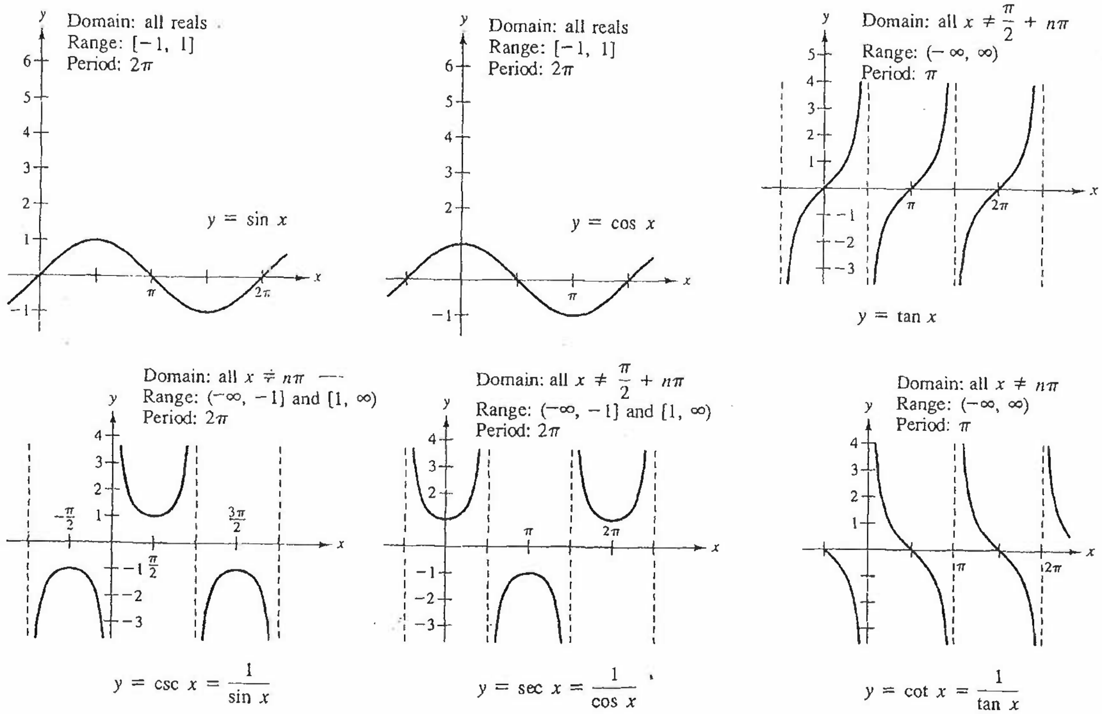


FIGURE 71
The graphs of the six trigonometric functions.

Note in Figure 71 that the maximum value of $\sin x$ and $\cos x$ is 1 and the minimum value is -1. The graphs of the functions $y = a \sin bx$ and $y = a \cos bx$ oscillate between $-a$ and a, and hence have an **amplitude** of $|a|$. Furthermore, because $bx = 0$ when $x = 0$ and $bx = 2\pi$ when $x = 2\pi/b$, it follows that the functions $y = a \sin bx$ and $y = a \cos bx$ each have a period of $2\pi/|b|$. Table 5 summarizes the amplitudes and periods for some types of trigonometric functions.

TABLE 5

Function	*Period*	*Amplitude*
$y = a \sin bx$ or $y = a \cos bx$	$\frac{2\pi}{\lvert b \rvert}$	$\lvert a \rvert$
$y = a \tan bx$ or $y = a \cot bx$	$\frac{\pi}{\lvert b \rvert}$	Not applicable
$y = a \sec bx$ or $y = a \csc bx$	$\frac{2\pi}{\lvert b \rvert}$	Not applicable

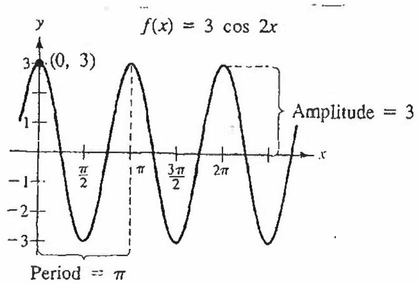


FIGURE 72

EXAMPLE 6 Sketching the Graph of a Trigonometric Function

Sketch the graph of $f(x) = 3 \cos 2x$.

Solution The graph of $f(x) = 3 \cos 2x$ has an amplitude of 3 and a period of $2\pi/2 = \pi$. Using the basic shape of the graph of the cosine function, sketch one period of the function on the interval $[0, \pi]$, using the following pattern.

Maximum: $(0, 3)$ Minimum: $\left(\frac{\pi}{2}, -3\right)$ Maximum: $[\pi, 3]$

By continuing this pattern, you can sketch several cycles of the graph, as shown in Figure 72.

The discussion of horizontal shifts, vertical shifts, and reflections given in Section 5 can be applied to the graphs of trigonometric functions, as illustrated in Example 7.

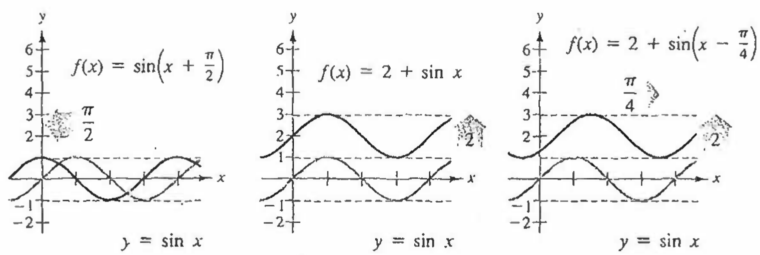
EXAMPLE 7 Shifts of Graphs of Trigonometric Functions

Sketch the graph of the following functions.

a. $f(x) = \sin\left(x + \dfrac{\pi}{2}\right)$ b. $f(x) = 2 + \sin x$ c. $f(x) = 2 + \sin\left(x - \dfrac{\pi}{4}\right)$

Solution

a. To sketch the graph of $f(x) = \sin(x + \pi/2)$, shift the graph of $y = \sin x$ to the left $\pi/2$ units, as shown in Figure 73(a).
b. To sketch the graph of $f(x) = 2 + \sin x$, shift the graph of $y = \sin x$ up 2 units, as shown in Figure 73(b).
c. To sketch the graph of $f(x) = 2 + \sin(x - \pi/4)$, shift the graph of $y = \sin x$ up 2 units and to the right $\pi/4$ units, as shown in Figure 73(c).



(a) Horizontal shift to the left (b) Vertical shift upward (c) Horizontal and vertical shift

FIGURE 73
Transformations of the graph of $y = \sin x$.

EXERCISES for Section 6

In Exercises 5 and 6, express the angles in radian measure as multiples of π and as decimals accurate to three decimal places.

5. a. 30° b. 150° c. 315° d. 120°

6. a. −20° b. −240° c. −270° d. 144°

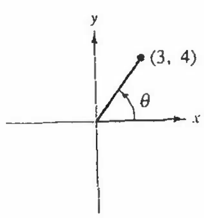
In Exercises 7 and 8, express the angles in degree measure.

7. a. $\dfrac{3\pi}{2}$ b. $\dfrac{7\pi}{6}$ c. $-\dfrac{7\pi}{12}$ d. −2.367

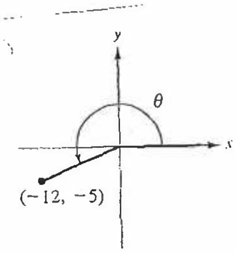
8. a. $\dfrac{7\pi}{3}$ b. $-\dfrac{11\pi}{30}$ c. $\dfrac{11\pi}{6}$ d. 0.438

In Exercises 13 and 14, determine all six trigonometric functions for the angle θ.

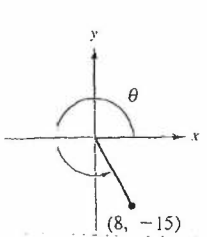
13. a.



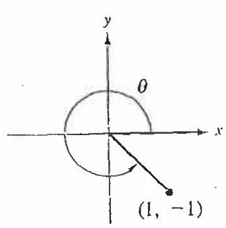
b.



14. a.



b.



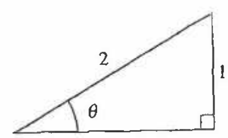
In Exercises 15 and 16, determine the quadrant in which θ lies.

15. a. $\sin\theta < 0$ and $\cos\theta < 0$
b. $\sec\theta > 0$ and $\cot\theta < 0$

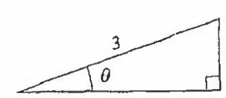
16. a. $\sin\theta > 0$ and $\cos\theta < 0$
b. $\csc\theta < 0$ and $\tan\theta > 0$

In Exercises 17–22, evaluate the trigonometric function.

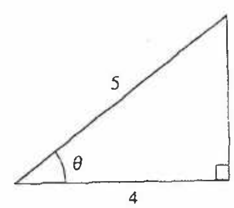
17. $\sin\theta = \frac{1}{2}$
$\cos\theta = ?$



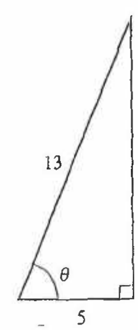
18. $\sin\theta = \frac{1}{3}$
$\tan\theta = ?$



19. $\cos\theta = \frac{4}{5}$
$\cot\theta = ?$



20. $\sec\theta = \frac{13}{5}$
$\csc\theta = ?$



In Exercises 23–26, evaluate the sine, cosine, and tangent of each angle *without* using a calculator.

23. a. 60°
b. 120°
c. $\frac{\pi}{4}$
d. $\frac{5\pi}{4}$

24. a. -30°
b. 150°
c. $-\frac{\pi}{6}$
d. $\frac{\pi}{2}$

25. a. 225°
b. -225°
c. $\frac{5\pi}{3}$
d. $\frac{11\pi}{6}$

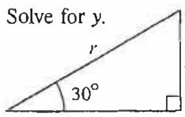
26. a. 750°
b. 510°
c. $\frac{10\pi}{3}$
d. $\frac{17\pi}{3}$

In Exercises 35–42, solve the equation for θ ($0 \le \theta < 2\pi$).

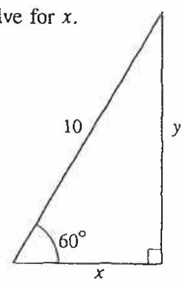
35. $2\sin^2\theta = 1$
36. $\tan^2\theta = 3$
37. $\tan^2\theta - \tan\theta = 0$
38. $2\cos^2\theta - \cos\theta = 1$
39. $\sec\theta\csc\theta = 2\csc\theta$
40. $\sin\theta = \cos\theta$
41. $\cos^2\theta + \sin\theta = 1$
42. $\cos\frac{\theta}{2} - \cos\theta = 1$

In Exercises 43–46, solve for x, y, or r as indicated.

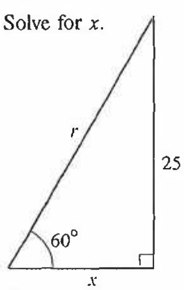
43. Solve for y.



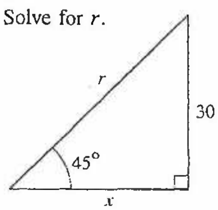
44. Solve for x.



45. Solve for x.



46. Solve for r.



47. *Airplane Ascent* An airplane leaves the runway climbing at 18° with a speed of 275 feet per second (see figure). Find the altitude of the plane after 1 minute.

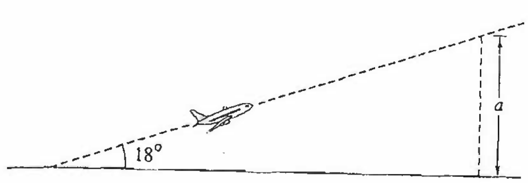
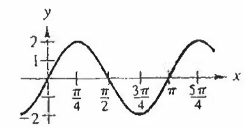


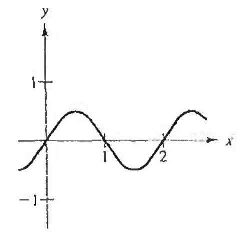
FIGURE FOR 47

In Exercises 51–54, determine the period and amplitude of each function.

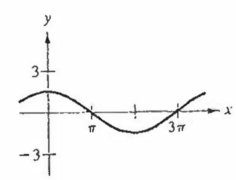
51. a. $y = 2\sin 2x$



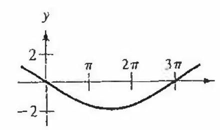
b. $y = \frac{1}{2}\sin\pi x$



52. a. $y = \frac{3}{2}\cos\frac{x}{2}$



b. $y = -2\sin\frac{x}{3}$



53. $y = 3\sin 4\pi x$
54. $y = \frac{2}{3}\cos\frac{\pi x}{10}$

In Exercises 55–58, find the period of the function.

55. $y = 5\tan 2x$
56. $y = 7\tan 2\pi x$
57. $y = \sec 5x$
58. $y = \csc 4x$

In Exercises 61–72, sketch the graph of the function.

61. $y = \sin\frac{x}{2}$
62. $y = 2\cos 2x$
63. $y = -\sin\frac{2\pi x}{3}$
64. $y = 2\tan x$
65. $y = \csc\frac{x}{2}$
66. $y = \tan 2x$
67. $y = 2\sec 2x$
68. $y = \csc 2\pi x$
69. $y = \sin(x + \pi)$
70. $y = \cos\left(x - \frac{\pi}{3}\right)$
71. $y = 1 + \cos\left(x - \frac{\pi}{2}\right)$
72. $y = 1 + \sin\left(x + \frac{\pi}{2}\right)$